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Solitary Waves in Liquids Having Non-Constant Density

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Part I

Liquids with Two Layers of Different Density

1. Introduction

The main purpose of these papers is to discuss what are called solitary waves in gravitating incompressible liquids of non-constant density. These are steady two-dimensional flows with a free surface in a channel with a horizontal bottom which extends to infinity in both directions. The essential characteristic of such waves is that they have a single crest while the vertical displacement of the free surface tends to the equilibrium level at infinity. For liquids of constant density the problem has a lengthy history which will be outlined a little later in this introduction.

Before recounting this history, however, it is of interest to examine the problem in a general way with reference to the underlying mathematical theory; first of all for the known case of a single layer having constant density. A velocity potential $\Phi(x,y)$ is to be found for the space occupied by the liquid. As is well known, this function is harmonic,

$$\Phi_{xx} + \Phi_{yy} = 0 \ , \tag{1.1}$$

and it satisfies the condition

$$\Phi_y(x,0) = 0 \tag{1.2}$$

at the bottom of the channel. At the free surface, characterized by the equation $y = y_s(x)$, (see Fig. 1.1) the boundary conditions are

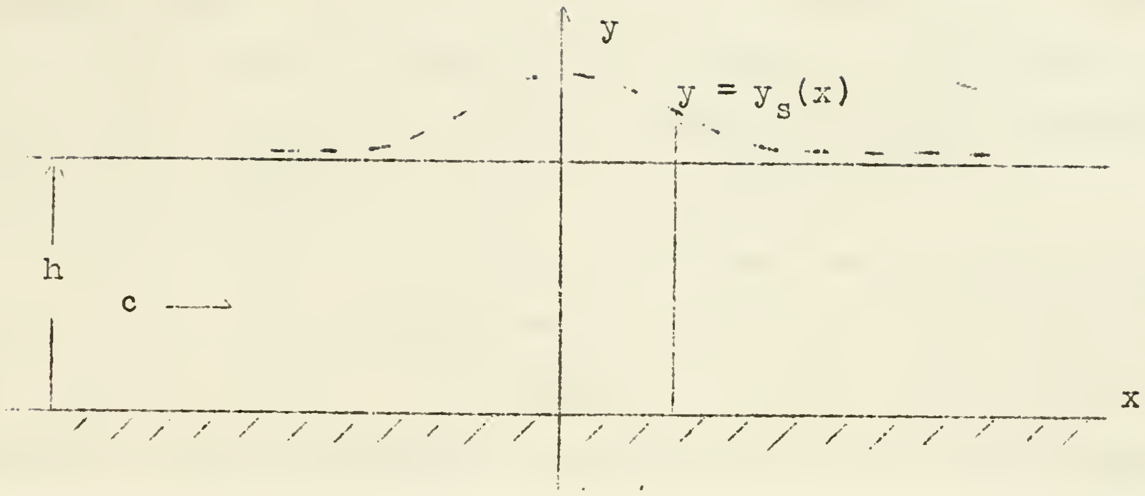

Fig. 1.1

$$\Phi_y(x,y_s) - y_{sx}\Phi_x(x,y_s) = 0 \,, \tag{1.3}$$

$$2gy_s + \Phi_x^2(x,y_s) + \Phi_y^2(x,y_s) = 2gh + c^2 \,. \tag{1.4}$$

The first of these conditions is the kinematic condition, the second is Bernoulli's law, with the pressure presumed zero at the free surface. At infinity the conditions are that $y_s \longrightarrow h$ and $\Phi_x \longrightarrow c$, a constant. Even though our main purpose is to discuss solitary waves, it has some point to discuss here also the case of waves which are periodic in x.

Because of the unknown position of the free surface and the nonlinear boundary conditions there, no frontal attack on the problem formulated above has been successful. However, procedures involving developments with respect to a small parameter have been employed to obtain flows in a neighborhood of the flows with constant horizontal velocity everywhere, and a horizontal free surface $y_s = h$. These constant flows are, as one sees immediately, exact solutions of the

nonlinear problem: $\Phi(x,y) \equiv cx$, $y_s(x) = h$ satisfy all the conditions (1.1) to (1.4), and the conditions at infinity. If the depth h is fixed, we have therefore a one-parameter family of exact solutions of the problem with the velocity c as parameter.

Two quite different procedures have been used to find solutions of the problem in the neighborhood of these flows. The first, and more direct, of these is a perturbation procedure with respect to the amplitude of the disturbance; this theory thus is in the category of small vibration theory of a continuum near an equilibrium position. One simply assumes for Φ and y_s developments of the form

$$\Phi(x,y) = cx + \varepsilon\phi_1(x,y) + \varepsilon^2\phi_2(x,y) + \ldots \quad , \tag{1.5}$$

$$y_s(x) = h + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \ldots \quad , \tag{1.6}$$

and inserts them in all of the basic equations, which then furnish a sequence of linear problems for the successive determination of the coefficients of the series. A consequence of this procedure is that the free surface conditions are to be satisfied at $y = h$, that is, at the equilibrium position of the free surface (as always in problems of small vibrations about an equilibrium position). For the lowest order terms the free surface conditions are

$$\phi_{1y}(x,h) - cy_{1x} = 0 \; , \tag{1.7}$$

$$gy_1 + c\phi_{1x}(x,h) = 0 \; . \tag{1.8}$$

The elimination of $y_1(x)$ gives

$$g\phi_{1y}(x,h) + c^2\phi_{1xx}(x,h) = 0 \; . \tag{1.9}$$

Of course, all functions $\phi_i(x,y)$ are harmonic functions in the strip $-\infty < x < \infty$, $0 < y < h$ subject to $\phi_{iy}(x,0) = 0$. Both the ϕ_i and y_i are assumed to be bounded at infinity. There exist periodic solutions of this homogeneous linear problem for $\phi_1(x,y)$, aside from the "trivial" solutions $\phi_1 = 0$, $y_1 = 0$, namely

$$\phi_1(x,y) = A \cosh my \cdot \sin m(x+\alpha) \ , \tag{1.10}$$

with A,α arbitrary constants, provided that

$$\frac{\tanh mh}{mh} = \frac{c^2}{gh} \ . \tag{1.11}$$

These solutions have been shown by Weinstein [22] to be the only solutions of the problem which are bounded at infinity. The condition (1.11) insures that the free surface condition (1.9) is satisfied. Once c and h are given, the equation (1.11) determines the wave number m or the wave length T from $T = 2\pi/m$. The equation (1.11) also displays the important dimensionless parameter c^2/gh, which fixes the character of the solutions. The solutions are quite different depending upon whether c^2/gh is less than, equal to, or greater than 1; and in fact the situation with respect to the linear problem is well known to be as follows (cf. for example Stoker [24], p. 207):

1. $c^2/gh < 1$. A periodic wave exists. For fixed h the wave-length T increases with c and tends to ∞ as $c^2/gh \longrightarrow 1$.

2. $c^2/gh \geq 1$. Only the trivial solution exists.

In the first case it has been proved by Struik [23], following the method of Levi-Civita [10] for infinite depth, that rigorous

periodic solutions of the exact nonlinear problem exist by showing, in effect, that the series (1.5) and (1.6) converge for ε sufficiently small. Thus the waves furnished by the linear theory are indeed the lowest order approximation in a convergent development with respect to amplitude. There are bifurcations of the solutions at every value of c with $c^2/gh < 1$. Indeed, more than one exact solution exists for these parameter values. In the second case no solutions of the nonlinear problem except the trivial solution can be found by this method--all of the ϕ_i and y_i would vanish.

It is worthwhile to state what happens if the problem for ϕ_1 and y_1 is made nonhomogeneous, say by inserting a small obstacle in the bed of the stream, or by applying a small nonzero pressure over a portion of the free surface to create a disturbance. In case 1) no unique solution can exist, clearly, since the solution (1.10) of the homogeneous problem can always be added; in this case a unique solution is picked out in practice by assuming a radiation condition at infinity, i.e., that the disturbance created dies out far upstream. In case 2), no radiation condition need be applied, since the homogeneous problem has only the trivial solution. However, in the special case $c^2/gh = 1$, the steady flow solution for ϕ_1 becomes unbounded at infinity in contradiction with the linearization. Indeed, as we shall have ample occasion to see, flows at and near the critical speed given by $c = \sqrt{gh}$, which is the upper bound for the speed of sinusoidal progressing waves in still water, are special in character and require a special treatment. However, L. Nirenberg has proved, in an unpublished paper, that rigorous solutions of the nonlinear problem exist both if $c < \sqrt{gh}$, and if $c > \sqrt{gh}$ when obstacles of small amplitude (i.e., of first order in ε) are put in the stream.

In addition to the steady flows it is also interesting to consider the linear theory of the unsteady flows which result when small obstacles are put in the stream with uniform flow at $t = 0$ and it is required to describe how the disturbance builds up as time goes on. If $c^2/gh \neq 1$, one finds as $t \to \infty$ the expected result that the solutions tend to the steady flows which were in question in the preceding paragraphs. However, if $c^2/gh = 1$, i.e., if the undisturbed flow has the critical speed, it is found (cf. [24]) that the linear theory yields a disturbance which becomes large everywhere without limit as $t \to \infty$. One would be tempted to say that the flow with critical speed is very unstable--the procedure formulated by the linear theory is, in fact, the standard small variation method for discussing the stability of a motion. The small oscillation criterion fails, however, in this case, it would seem, from both the mathematical and experimental points of view, since the wave called the solitary wave exists in the vicinity of the critical speed. From the point of view of observations and of laboratory experiments, this wave seems to be extremely stable, and it also seems rather easy to create it.

Recently, Friedrichs and Hyers [16] have proved rigorously the mathematical existence of a solitary wave solution of the full nonlinear problem. The method they use has been referred to above as the second of two basic procedures. It is not a method of perturbation with respect to amplitude. Instead, it is an iteration procedure based on the small parameter $\sigma = c^2/gh - 1$, and it therefore furnishes solutions valid near the uniform flow at critical speed. This procedure, however, requires that the dependent and independent

variables be stretched in ways which depend on σ, and also in a manner which distinguishes the horizontal from the vertical direction. To lowest order, however, it furnishes the same result as a formal perturbation procedure obtained by setting up power series in σ (which are set up after the stretching process, of course); this was carried out earlier by Keller[15]. In fact, to lowest order, the theory is identical with the approximate theory called the nonlinear long wave or shallow water theory. The result in the lowest order of approximation is essentially the same as the well-known much earlier results of Boussinesq [3], Rayleigh [4], and others.

It is highly unlikely that perturbation series in powers of the stretching parameter σ would converge; such series seem certain to have at best asymptotic significance since the stretching process has the effect of making σ a coefficient of some of the highest derivatives occurring in the problem, so that boundary layer effects are to be expected in the limit, and the perturbation procedure seems not likely to be of the regular type. In this paper, formal series in the parameter σ are used to obtain the lowest order terms.

In this manner one can find solitary waves of arbitrarily small amplitude; but these waves cannot be obtained by perturbation series in the amplitude in the way described above, as was already remarked: the coefficients of such series, on the basis of the uniqueness theorem of Weinstein, would all vanish. Thus the solitary wave is not analytic in the amplitude.

Although only passing remarks are made about so called cnoidal wave solutions in this paper, it is perhaps worthwhile to mention them briefly here. In addition to the solitary wave, there exist

also periodic waves in the neighborhood of the critical speed. These waves are obtained by the same means, described above, which lead to the solitary wave, and Littman [18] has proved the existence of such waves by using a method similar to that of Friedrichs and Hyers. For small departures from the critical speed these waves have large wave lengths and in the limit as the wave length goes to infinity they tend to the solitary wave.

The object of this paper is to treat problems of the same general types as those first discussed, but for liquids of non-constant density. In doing so attention is centered mostly on solutions of the type of the solitary wave or cnoidal wave. No convergence proofs are attempted; lowest order terms only in formal perturbation schemes are obtained. However, it seems highly probable that these solutions are indeed the correct lowest order terms in the development of exact solutions of the nonlinear problem with respect to the appropriate parameters.

In Part I, the case of a fluid with two layers each of constant density, with a free surface on top, is treated. Both the linear theory of waves of small amplitude, and a nonlinear theory of the same kind as the long wave or shallow water theory are employed. The linear theory furnishes the important result in this case that there are <u>two</u> critical speeds. These speeds are defined as the limiting speeds of waves whose wave length tends to infinity. For layers of equal thickness $h/2$, for example, but different densities δ, $\underline{\triangle}$ such that if δ is the density of the lowest layer $\rho = \underline{\triangle}/\delta \leq 1$, these speeds are given by

$$c^2 = \frac{gh}{2} [1 \pm \sqrt{\rho}] \text{ .}$$

Thus if the densities are nearly the same the larger critical speed is near $\sqrt{gh}$ as one might expect, and the smaller is near zero. From the discussion above one might also expect that two solitary waves would exist with speeds near the critical speeds. That turns out to be the case. The two critical speeds distinguish two types of solitary waves. One type has characteristics similar to the ordinary solitary wave. The stream lines for this type are lines of elevation above the undisturbed stream lines and they have a maximum amplitude at the free surface. The other type tends to be hidden from observation at the free surface. The waves are internal with a maximum amplitude at the interface much larger than the amplitude at the free surface. Depending on a value of a function of the ratio of the depths and the ratio of the densities of the layers, the internal wave at the interface may be either a wave of depression or elevation. Cnoidal waves near the two critical speeds can also be determined.

In Part II, where fluids with a vertical density distribution which decreases exponentially going upward are treated, it is found that the linear theory for waves of small amplitude furnishes an infinite spectrum of critical speeds. These speeds form a discrete set with 0 as the only limit point. One is led to conjecture that there would be infinitely many solitary (and also cnoidal) waves associated with the critical speeds and that the extrema of each of these waves would be associated with different levels in the fluid which would increase in number and begin nearer the bottom as the corresponding critical speed is taken smaller. These conjectures all prove to be true.

We turn now to a discussion of the earlier history of this subject. The main object of the account which follows is to briefly indicate the basic methods which have been used in the development of the theory of solitary and cnoidal waves, and to point to the fundamental theoretical papers which to our knowledge are concerned primarily with solitary or cnoidal waves. There are papers which are concerned with the application of the solitary wave theory to ocean waves and atmospheric waves, and there are others which report results of laboratory experiments; some of these are listed toward the end of the bibliography.

The name solitary wave is due to Scott Russell [1] who presented the first scientific report (1844) about such a wave. Scott Russell was an engineer who studied the solitary wave in connection with an investigation designed to ascertain the feasibility of operating steam driven craft in the Edinburgh and Glasgow canal. His observations led Russell to adopt the empirical formula

$$c = \sqrt{g(h+a)}$$

for the velocity c of the solitary wave in terms of g, the acceleration due to gravity; h the undisturbed depth of the channel; and a, the amplitude of the wave which is not necessarily small. This formula was confirmed by the more precise experiments conducted by Bazin [2] in 1865. The first mathematical theories which gave approximate descriptions of the profile and speed of the solitary wave were presented independently by Boussinesq [3] (1871) and Rayleigh [4] (1876). Rayleigh's analysis proceeds from the expansion

of a time independent complex velocity potential in powers of the vertical coordinate. Boussinesq used the first terms in a similar expansion of a time dependent velocity potential and then supplemented his analysis with various physical assumptions in addition to those implied by the basic hydrodynamical theory.

McCowan [5] (1891) introduced a complex velocity potential which gave a second order approximation to the speed of the solitary wave. Korteweg and DeVries [6] (1895) used a modification of Rayleigh's method to show that a general class of long waves of finite amplitude and permanent type may exist. They found waves described by the Jacobian elliptic function $cn(\xi,k)$. The existence of these periodic waves which travel without change of form was first indicated by Boussinesq [7] (1891). Korteweg and DeVries called these waves <u>cnoidal</u> waves and showed that the limit of a cnoidal wave, whose wave length approaches infinity, is a solitary wave. Gwyther [8] (1900) and Levi-Civita [9] (1907) presented refinements and systemizations of Rayleigh's method.

In 1925 Levi-Civita [10] proved the mathematical existence of two-dimensional periodic waves of finite amplitude and permanent type in water of infinite depth. Levi-Civita used the velocity potential function and the stream function as independent variables. With these variables the problem was reduced to finding a function analytic in a known domain and subject to a certain boundary condition nonlinear over part of the boundary. Weinstein [11] (1926) used Levi-Civita's method to improve the known approximations to the speed of the solitary wave. A numerical mistake in Weinstein's second order formula has been corrected by Long [20]. Davies [12]

(1951) investigated waves of finite amplitude by modifying the non-linear condition which appears in the Levi-Civita presentation; and then Packham [13] (1952), using Davies' technique, studied the solitary wave and found a formula for the speed similar to the one found by McCowan.

In an appendix to a paper by Stoker [14] (1948) Friedrichs gave a systematic perturbation method designed to produce the conventional non-linear shallow water theory in the first order approximation, and to exhibit a parameter which determines the accuracy of this and higher order approximations. Friedrichs' method is characterized by the introduction of a dimensionless parameter σ which he used to stretch some of the variables; and the assumption that all quantities can be expanded in powers of σ. Keller [15] (1948) used Friedrichs' method to show that in the case of steady motion, the second order approximation gives cnoidal and solitary waves. Keller developed a systematical procedure for the theoretical study of these waves and found results in essential agreement with those found by Rayleigh, Boussinesq, and Korteweg and DeVries. Later Friedrichs and Hyers [16] (1954) used Friedrichs' method to prove the mathematical existence of the solitary wave. In [16] Lavrentiev [17] (1946), who used the Levi-Civita approach, is credited as being the first to give a proof of the existence of the solitary wave. Littman [18] (1957) used a method similar to the one in [16] to prove the existence of cnoidal waves near critical speed.

In each of the papers mentioned above it was assumed that the density of the medium is constant. In recent years, there has been a growing interest in solitary waves in a stratified medium. The

interest stems from the surmise that solitary waves may play a considerable role in certain meteorological phenomena and phenomena in other fields where density changes are important. Keulegan [19] (1953) used Boussinesq's method to investigate the characteristics of internal solitary waves at the interface of a two-layer medium bounded by rigid planes at both the top and bottom surfaces. Long [20] (1956) used Rayleigh's method to find a fourth order approximation to the speed of the ordinary solitary wave, and in the same paper he verifies some of Keulegan's results and gives an second order approximation to the internal solitary wave of a two-fluid system, again bounded by rigid planes above and below. Abdullah [21] (1955) used Keller's procedure to discuss the solitary wave at the interface of a two-layer medium for which it was assumed that the hydrostatic law holds in the upper layer.

In the present part a variant of Friedrichs' method is used to obtain solitary waves in a two-layer medium with a free surface at the top of the upper layer. In Part II we will present results which have been found for solitary waves in a medium in which there is a continuous variation of density. In addition to results already mentioned, we present in Part I formulas for the velocities of the two kinds of waves in terms of the maximum amplitudes, and expressions for the streamlines throughout the medium as well as expressions for the velocity components and pressure of the flow.

2. Formulation of the Problem

In order to study two-dimensional waves of permanent type in a two-layer medium of immiscible liquids let us suppose that a cross section of the medium in the equilibrium state fills an infinite horizontal strip. The lower layer whose density is constant and equal to δ is supposed to be supported by a rigid bottom; and the upper layer, whose density is constant and equal to $\triangle$ such that $\triangle \leq \delta$, is supposed to have a free surface, i.e., a surface on which the pressure p is zero and there are no geometric constraints. Let us suppose that a disturbance of some kind in the medium initially at rest has created a wave of permanent type which moves to the left with velocity c; and that the character of this wave is observed from a coordinate system which moves with the wave. The x-axis of this system is taken to coincide with the bottom, and the vertical y-axis is chosen so as to pass through a crest of the wave. With respect to these axes the wave is stationary and the velocity of the medium at infinity is c. As shown in the Fig. 2.1, let us take the depths, or the mean depths, of the lower and upper layers at infinity to be h and H respectively.

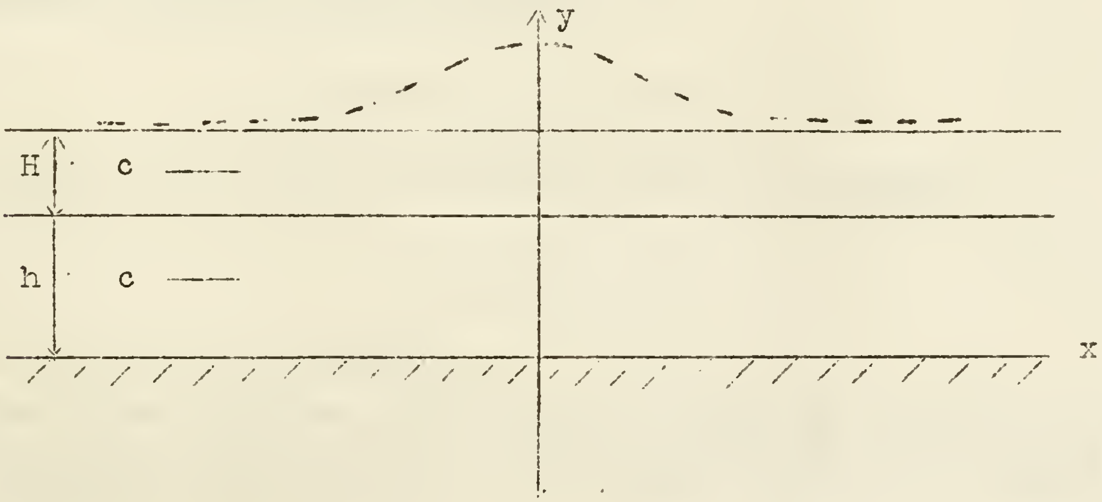

Fig. 2.1

The steady state equations which have to be satisfied for the lower layer are

$$\delta(\tilde{u}\tilde{u}_x+\tilde{v}\tilde{u}_y) = -\tilde{p}_x ,$$
$$\delta(\tilde{u}\tilde{v}_x+\tilde{v}\tilde{v}_y) = -\delta g - \tilde{p}_y ,$$
$$\tilde{u}_x+\tilde{v}_y = 0 ,$$
$$\tilde{u}\tilde{s}_x+\tilde{v}\tilde{s}_y = 0 ,$$
$$\tilde{\zeta} = \tilde{v}_x-\tilde{u}_y ,$$

where $\tilde{u},\tilde{v}$ are the velocity components; $\tilde{p}$ is the pressure; g is the acceleration due to gravity; $\tilde{s}(x,y) = 0$ is the equation of a boundary curve; and $\tilde{\zeta}$ is the vorticity. The domain in which the solution is to be found is partly unknown at the outset. If the density δ is replaced by $\triangle$ we have the equations for the upper layer. Hereafter, quantities which pertain to the upper layer, and which need to be distinguished from similar quantities in the lower layer, will be denoted by capital letters.

If the stream function (*) is $\psi(x,y)$ so that

$$\tilde{u} = \psi_y(x,y) \qquad\qquad \tilde{v} = -\psi_x(x,y)$$

the strema lines are given implicitly by

$$\psi(x,y) = \gamma = \text{const.}$$

The function $\psi(x,y)$ can be chosen so that for the bottom

$$\gamma = \psi(x,0) = 0$$

(*) The stream function, rather than the velocity potential, is introduced because it is convenient here, and more or less indispensable for the case of continuous variation in density to be treated in Part II.

and then the value of γ which determines the interface can be taken to be

$$\gamma = ch ,$$

while for the free surface we may write

$$\gamma = c(h+H) = ch(1+r)$$

where $r = H/h$. It is assumed that to each value of γ there corresponds one and only one stream line which is given by solving $\gamma = \psi(x,y)$ for y namely,

$$y = \bar{f}(x,\gamma) .$$

The function $\bar{f}(x,\gamma)$ will be called the stream line function. This assumption is reasonable since flows near uniform flows are the only ones considered.

Let us introduce x and γ as independent variables and use the notation

$$\tilde{u}(x,y) = \tilde{u}[x,\bar{f}(x,\gamma)] = \bar{u}(x,\gamma)$$

and a similar bar notation for the other transformed circumflex quantities. Then, for example

$$\tilde{u}_x(x,y) = \bar{u}_x+\bar{u}_\gamma \frac{\partial\gamma}{\partial x} = \bar{u}_x-\bar{v}\bar{u}_\gamma$$
$$\tilde{u}_y(x,y) = \bar{u}_\gamma \frac{\partial\gamma}{\partial y} = \bar{u}\bar{u}_\gamma$$

and using these relations the momentum equations for the lower layer become

$$\delta\bar{u}\bar{u}_x = -\bar{p}_x+\bar{v}\bar{p}_\gamma ,$$
$$\delta\bar{u}\bar{v}_x = -\delta g - \bar{u}\bar{p}_\gamma ,$$

the continuity equation becomes

$$\bar{u}_x - \bar{v}\bar{u}_\gamma + \bar{u}\bar{v}_\gamma = 0 \ ,$$

and the vorticity is given by

$$\bar{\zeta} = \bar{v}_x - \bar{v}\bar{v}_\gamma - \bar{u}\bar{u}_\gamma \ .$$

The kinematic condition at a boundary curve becomes

$$\bar{u}(\bar{s}_x - \bar{v}\bar{s}_\gamma) + \bar{v}\bar{u}\bar{s}_\gamma = 0$$

or

$$\bar{u}\bar{s}_x = 0$$

but this of course is automatically satisfied since for a boundary curve we have $\bar{s}(x,\gamma) = 0$ where γ is a constant corresponding to the boundary stream line. The domain for which the differential equations must be satisfied is now the strip $-\infty < x < \infty ; \ 0 < \gamma < ch$. By working with the stream function in this way the advantage of a fixed domain for the independent variables is achieved.

Instead of using the dependent variables $\bar{u}$, $\bar{v}$ and $\bar{p}$ it is more convenient for our purposes to use $\bar{u}$, $\bar{f}$ and $\bar{p}$ as the dependent variables. For these variables we have

$$\bar{f}_\gamma = \frac{1}{\psi_y} = \frac{1}{\bar{u}} \qquad \qquad \bar{f}_x = \frac{-\psi_x}{\psi_y} = \frac{\bar{v}}{\bar{u}}$$

and it can easily be verified that the continuity equation

$$\bar{u}_x - \bar{v}\bar{u}_\gamma + \bar{u}\bar{v}_\gamma = 0$$

is automatically satisfied. We do not expect $\bar{u}$ to vanish since the flow is near the uniform flow with velocity $c \neq 0$. The momentum equations now take the form

$$\delta \bar{u}_x = -\bar{f}_\gamma \bar{p}_x + \bar{f}_x \bar{p}_\gamma$$

$$\delta(\bar{u}\bar{f}_{xx} + \bar{u}_x \bar{f}_x) = -\delta g \bar{f}_\gamma - \bar{p}_\gamma$$

and with these we have

$$\bar{u}\bar{f}_\gamma = 1 ,$$

as a replacement for the continuity equation. The vorticity is given by

$$\bar{\zeta} = \frac{\partial}{\partial x}\left(\frac{\bar{f}_x}{\bar{f}_\gamma}\right) - \frac{1}{2}\frac{\partial}{\partial \gamma}\left(\frac{1+\bar{f}_x^2}{\bar{f}_\gamma^2}\right) .$$

Finally, if we introduce the dimensionless variables

$$\xi = \frac{x}{h} \qquad \eta = \frac{\gamma}{ch} \qquad u = \frac{\bar{u}}{c} \qquad U = \frac{\bar{U}}{c}$$

$$v = \frac{\bar{v}}{c} \qquad V = \frac{\bar{V}}{c} \qquad p = \frac{\bar{p}}{\delta c^2} \qquad P = \frac{\bar{P}}{\Delta c^2}$$

$$f = \frac{\bar{f}}{h} \qquad F = \frac{\bar{F}}{h} \qquad \lambda = \frac{gh}{c^2} \qquad \rho = \frac{\Delta}{\delta}$$

$$\zeta = \frac{h}{c}\bar{\zeta}$$

we find that the differential equations become

$$\begin{aligned} u_\xi &= - f_\eta p_\xi + f_\xi p_\eta , \\ u f_{\xi\xi} + u_\xi f_\xi &= - \lambda f_\eta - p_\eta , \\ u f_\eta &= 1 . \end{aligned} \tag{2.1}$$

They hold in the strip $-\infty < x < \infty$; $0 < \eta < 1$. In the upper layer the equations are, in the same way:

$$
\begin{aligned}
U_\xi &= - F_\eta P_\xi + F_\xi P_\eta \\
UF_{\xi\xi} + U_\xi F_\xi &= - \lambda F_\eta - P_\eta \\
UF_\eta &= 1 ,
\end{aligned}
\tag{2.2}
$$

and they hold in the strip $-\infty < x < \infty$; $1 < \eta < 1+r$. The boundary condition at the bottom is

$$f(x,0) = 0 .$$

The boundary conditions at the interface are

$$f(x,1) = F(x,1) ; \quad p(x,1) = \rho P(x,1) .$$

These conditions state that the interface is a stream line and that the pressure is continuous along it. Since the pressure at the free surface is zero the boundary condition there is

$$P(x,1+r) = 0 .$$

For irrotational motion, which we assume, the vorticity must be zero in each layer and this means that both f and F must satisfy

$$\frac{\partial}{\partial\xi}\left(\frac{f_\xi}{f_\eta}\right) - \frac{1}{2}\frac{\partial}{\partial\eta}\left(\frac{1+f_\xi^2}{f_\eta^2}\right) = 0$$

or

$$(1+f_\xi^2)f_{\eta\eta} + f_\eta^2 f_{\xi\xi} - 2f_\xi f_\eta f_{\xi\eta} = 0 . \tag{2.3}$$

The equation which expresses Bernoulli's law is found by eliminating the horizontal velocity component from equations (2.1), (2.2) and integrating; for example:

$$\frac{\partial}{\partial\xi}\left(\frac{1}{f_\eta}\right) = - f_\eta p_\xi + f_\xi p_\eta$$

$$\frac{\partial}{\partial\xi}(uf_\xi) = \frac{\partial}{\partial\xi}\left(\frac{f_\xi}{f_\eta}\right) = - \lambda f_\eta - p_\eta$$

$$\frac{\partial}{\partial\xi}\left(\frac{1}{f_\eta}\right) + f_\xi \frac{\partial}{\partial\xi}\left(\frac{f_\xi}{f_\eta}\right) = - f_\eta p_\xi - \lambda f_\xi f_\eta$$

$$\frac{1}{f_\eta}\frac{\partial}{\partial\xi}\left(\frac{1}{f_\eta}\right) + \frac{f_\xi}{f_\eta}\frac{\partial}{\partial\xi}\left(\frac{f_\xi}{f_\eta}\right) + p_\xi + \lambda f_\xi = 0$$

$$\frac{1}{2}\left(\frac{1+f_\xi^2}{f_\eta^2}\right) + p + \lambda f = C(\eta) \ .$$

For irrotational flow $C(\eta)$ is a constant.

3. The Linearized Equations. Critical Speeds

For the case of a steady parallel flow of velocity c in each layer we have

$$\begin{array}{ll} u = 1 & U = 1 \\ f = \eta & F = \eta \\ p = \lambda(1+\rho r-\eta) & P = \lambda(1+r-\eta) \ . \end{array}$$

We consider the wave motion to be a small disturbance superposed on this state of constant flow. We therefore write

$$\begin{array}{ll} u = 1+u^* & U = 1+U^* \\ f = \eta+f^* & F = \eta+F^* \\ p = \lambda(1+\rho r-\eta)+p^* & P = \lambda(1+r-\eta)+P^* \ . \end{array}$$

Then if we substitute these quantities in (2.1), (2.2), (2.3) and assume the disturbance so small that second order products involving the starred quantities can be neglected we find the linear equations

$$u^*_\xi = -p^*_\xi - \lambda f^*_\xi \qquad U^*_\xi = -P^*_\xi - \lambda F^*_\xi$$

$$f^*_{\xi\xi} = -\lambda f^*_\eta - p^*_\eta \qquad F^*_{\xi\xi} = -\lambda F^*_\eta - P^*_\eta$$

$$u^* + f^*_\eta = 0 \qquad U^* + F^*_\eta = 0$$

subject to the boundary conditions

$$f^*(\xi,0) = 0$$

$$f^*(\xi,1) = F^*(\xi,1) \qquad p^*(\xi,1) = \rho P^*(\xi,1)$$

$$P^*(\xi,1+r) = 0 \, .$$

For irrotational motion f and F must satisfy

$$\text{(3.1)} \qquad \begin{aligned} f^*_{\xi\xi} + f^*_{\eta\eta} &= 0 \qquad & -\infty < \xi < \infty, \quad 0 < \eta < 1 \\ F^*_{\xi\xi} + F^*_{\eta\eta} &= 0 \qquad & -\infty < \xi < \infty, \quad 1 < \eta < 1+r \end{aligned}$$

and the boundary conditions

$$f^*(\xi,0) = 0$$

$$f^*(\xi,1) = F^*(\xi,1) \qquad \lambda f^*_\xi(\xi,1) - f^*_{\xi\eta}(\xi,1) = \rho[\lambda F^*_\xi(\xi,1) - F^*_{\xi\eta}(\xi,1)]$$

$$F^*_{\eta\xi}(\xi,1+r) = \lambda F^*_\xi(\xi,1+r) \, .$$

The following functions are solutions of (3.1) which satisfy the boundary conditions at $\eta = 0$ and $\eta = 1$:

$$\text{(3.2)} \qquad \begin{aligned} f^* &= A \sin(\nu\xi + B) \sinh \nu \\ F^* &= \frac{A}{\rho\nu} \sin(\nu\xi + B) \begin{bmatrix} \nu \sinh \nu\eta - \nu(1-\rho) \sinh \nu \cosh \nu(1-\eta) \\ + \lambda(1-\rho) \sinh \nu \sinh \nu(1-\eta) \end{bmatrix} \end{aligned}$$

with ν, A, and B any constants.

There are solutions which correspond to the special value $\nu = 0$ of ν and which satisfy the boundary conditions at $\eta = 0$ and $\eta = 1$. They are:

$$f = (a\xi+b)\eta$$

$$F = \frac{(a\xi+b)}{\rho}\,[(1-\lambda+\lambda\rho)\eta+(1-\rho)(\lambda-1)]\ .$$

If a is not zero, these solutions give stream lines which depart as far as we please from $f = \eta$; $F = \eta$ and therefore they cannot be admitted in the linear theory.

If the functions in (3.2) are to satisfy the boundary condition at $\eta = 1+r$ for $\nu \neq 0$ one must have $(1+\rho \tanh \nu \tanh \nu r)\nu^2 - (\tanh \nu + \tanh \nu r)\lambda\nu+\lambda^2(1-\rho) \tanh \nu \tanh \nu r = 0$ which can be written

$$(3.3) \qquad (1+\rho \tanh \nu \tanh \nu r)\left[\frac{\nu}{\lambda} - a_1(\nu)\right]\left[\frac{\nu}{\lambda} - a_2(\nu)\right] = 0$$

where

$$a_1(\nu), a_2(\nu) = \frac{(\tanh\nu+\tanh\nu r) \pm \sqrt{(\tanh \nu - \tanh \nu r)^2 + 4\rho\tanh\nu\tanh\nu r[1-(1-\rho)\tanh\nu\tanh\nu r]}}{2(1+\rho \tanh \nu \tanh \nu r)}$$

Equation (3.3) determines the wave number in terms of the ratio of the depths of the layers $r = H/h$, the ratio of the densities $\rho = \triangle/\delta$ and the speed, since $\lambda = gh/c^2$. If T is the wave length of the sinusoidal waves, $\nu = 2\pi/T$; it is seen that (3.3) gives two speeds associated with a prescribed wave length. The limiting speeds corresponding to waves whose wave lengths approach infinity--this was one of the ways used to characterize critical speeds in the single

layer case--are given in this case by

$$(3.4)\qquad \begin{aligned} \frac{1}{\lambda_1} &= \frac{c_1^2}{gh} = \mathop{\mathcal{L}}_{\nu\to 0} \frac{\alpha_1(\nu)}{\nu} = \frac{1+r+\sqrt{(1-r)^2+4\rho r}}{2} \ , \\ \frac{1}{\lambda_2} &= \frac{c_2^2}{gh} = \mathop{\mathcal{L}}_{\nu\to 0} \frac{\alpha_2(\nu)}{\nu} = \frac{1+r-\sqrt{(1-r)^2+4\rho r}}{2} \ , \end{aligned}$$

as one readily sees. For still another reason, which will appear below, these speeds c_1 and c_2 are called critical speeds. It can be seen that the critical speeds satisfy the equation

$$(3.5)\qquad \lambda^2(1-\rho)r-(1+r)\lambda+1 = 0 \ .$$

Conversely, if λ corresponds to one of the critical speeds it is found that the left-hand side of (3.3) will contain the factor ν^4. From this we can see that the linear theory is inapplicable to non-homogeneous problems in which the speed is critical, just as is known for the case of one layer. To see this, consider a pressure disturbance which moves with velocity c over the free surface of the two-layer medium. The solution for the stream lines in the upper layer, for example, is

$$F^* = \frac{1}{2\pi\rho} \int_L \frac{\overline{P}^* e^{-i\xi\nu} \begin{bmatrix} \nu\sinh\nu\eta+\nu(\rho-1)\sinh\nu\cosh\nu(\eta-1) \\ +\ \lambda(\rho-1)\sinh\nu\sinh\nu(\eta-1) \end{bmatrix} d\nu}{\lambda^2\cosh\nu\cosh\nu r(1+\rho\tanh\nu\tanh\nu r)[\frac{\nu}{\lambda}-\alpha_1(\nu)][\frac{\nu}{\lambda}-\alpha_2(\nu)]}$$

where $\overline{P}^*$ is the Fourier transform of the pressure at the free surface; and L is a path along the real axis except for semi-circles, which avoid the zeros of the denominator, and are situated so that the

disturbance dies out upstream. Now if λ corresponds to one of the critical speeds the denominator in the integrand, which contains the left-hand side of (3.3), will have a fourth order zero at $\nu = 0$, while the numerator has a second order zero there. The effect of this is that when the path is deformed to the other side of the real axis in order to discuss the disturbance far downstream, the contribution from the second order pole at the origin will give stream lines which depart arbitrarily far from the lines of constant flow. This kind of behavior is inadmissible in the linear theory--the linear theory fails at a critical speed, and we are forced to turn to a nonlinear formulation. This is all just as it was for one layer. We are thus led to expect that the two critical speeds are speeds near which solitary waves will occur.

4. Waves Near the Two Critical Speeds

The non-linear equations (2.1) and (2.2) can be treated in several different ways. One way is to eliminate the velocities and pressures so as to obtain the equations which $f(\xi,\eta)$ and $F(\xi,\eta)$ must satisfy and then deal directly with these equations subject to the boundary conditions expressed in terms of f and F only. For irrotational motion these functions must satisfy

$$
(4.1) \qquad \begin{aligned}
&(1+f_\xi^2)f_{\eta\eta}+f_\eta^2 f_{\xi\xi}-2f_\xi f_\eta f_{\xi\eta} = 0 \ , && 0 < \eta < 1 \\
&(1+F_\xi^2)F_{\eta\eta}+F_\eta^2 F_{\xi\xi}-2F_\xi F_\eta F_{\xi\eta} = 0 \ , && 1 < \eta < 1+r \ .
\end{aligned}
$$

These equations could be analyzed by Rayleigh's method which involves expanding f and F in powers of the depth variable η. Another way to proceed, along the lines used by Friedrichs [14], is to stretch the horizontal variable in (4.1) by introducing the new variable $\sqrt{\varepsilon}\,\xi = \sigma$, and then assume that

$$f = \sum_{k=0}^{\infty} f_k(\sigma,\eta)\varepsilon^k \qquad\qquad F = \sum_{k=0}^{\infty} F_k(\sigma,\eta)\varepsilon^k .$$

The authors used both of these methods in preliminary work. Each method, although effective, has certain technical disadvantages. The method which follows seems to be best.

In order to investigate a wave which moves with a speed such that $\lambda = gh/c^2$ is near some value λ', we insert λ' in the equations (2.1) and (2.2) so that the difference $\lambda-\lambda'$ appears. In other words consider the equations

$$\begin{aligned} u_\xi &= -\,f_\eta p_\xi + f_\xi p_\eta \\ uf_{\xi\xi} + u_\xi f_\xi &= (\lambda'-\lambda)f_\eta - \lambda' f_\eta - p_\eta \qquad\qquad 0 < \eta < 1 \\ uf_\eta &= 1 \end{aligned}$$

and the corresponding set for the upper layer. If we write

$$\lambda' - \lambda = \varepsilon$$

and introduce the new variable

$$\sigma = \xi\sqrt{\varepsilon}$$

the equations become

$$
(4.2) \qquad
\begin{aligned}
u_\sigma &= - f_\eta p_\sigma + f_\sigma p_\eta \\
\varepsilon(u f_{\sigma\sigma} + u_\sigma f_\sigma) &= \varepsilon f_\eta - \lambda f_\eta - p_\eta \\
u f_\eta &= 1
\end{aligned}
\qquad 0 < \eta < 1
$$

and

$$
(4.3) \qquad
\begin{aligned}
U_\sigma &= - F_\eta P_\sigma + F_\sigma P_\eta \\
\varepsilon(U F_{\sigma\sigma} + U_\sigma F_\sigma) &= \varepsilon F_\eta - \lambda F_\eta - P_\eta \\
U F_\eta &= 1
\end{aligned}
\qquad 0 < \eta < 1+r.
$$

Since the vorticity is assumed to be zero we also have

$$
(4.4) \qquad
\begin{aligned}
f_{\eta\eta} + \varepsilon[f_\sigma^2 f_{\eta\eta} + f_\eta^2 f_{\sigma\sigma} - 2 f_\sigma f_\eta f_{\sigma\eta}] &= 0 \\
F_{\eta\eta} + \varepsilon[F_\sigma^2 F_{\eta\eta} + F_\eta^2 F_{\sigma\sigma} - 2 F_\sigma F_\eta F_{\sigma\eta}] &= 0 .
\end{aligned}
$$

Let us assume now that all of the quantities in the new variables can be expanded in integral powers of ε, that is, we write

$$
(4.5) \qquad
\begin{aligned}
f(\sigma,\eta,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k f_k(\sigma,\eta) \qquad u = \ldots \qquad p = \ldots \\
F(\sigma,\eta,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k F_k(\sigma,\eta) \qquad U = \ldots \qquad P = \ldots
\end{aligned}
$$

where the k^{th} order approximations are subject to the boundary conditions

$$
(4.6) \qquad
\begin{gathered}
f_k(\sigma,0) = 0 , \\
f_k(\sigma,1) = F_k(\sigma,1) , \qquad p_k(\sigma,1) = \rho P_k(\sigma,1) , \\
P_k(\sigma,1+r) = 0 .
\end{gathered}
$$

As a consequence of the assumption (4.5) the vertical velocity in the lower layer, for example, must have the expansion

$$
(4.7)\qquad
\begin{aligned}
v &= \frac{\bar{v}}{c} = \frac{\bar{u}\bar{f}_x}{c} = uf_\xi = \sqrt{\varepsilon}\, uf_\sigma \\
&= \sqrt{\varepsilon}\, [u_o+\varepsilon u_1+\ldots][f_{o\sigma}+\varepsilon f_{1\sigma}+\ldots] \\
&= \sqrt{\varepsilon}\, [u_o f_{o\sigma}+\varepsilon(u_1 f_{o\sigma}+u_o f_{1\sigma})+\ldots] \,.
\end{aligned}
$$

The idea is to substitute these expansions in (4.2), (4.3), (4.4) and the boundary conditions, and equate coefficients of like powers of ε. This will lead to a sequence of problems which are to be solved. The expectation is that the analysis will show automatically that ℓ must correspond to one of the critical speeds, and thus be equal to either λ_1 or λ_2.

We observe also that the parameter ε multiplies highest order derivatives in various equations, thus indicating that the perturbation is not of the regular type. Rather, an asymptotic expansion is the likely thing to expect.

The equations for the zeroth order approximation are

$$
(4.8)\qquad
\left.\begin{aligned}
u_{o\sigma} &= f_{o\sigma}p_{o\eta}-f_{o\eta}p_{o\sigma} \\
0 &= \ell f_{o\eta}+p_{o\eta} \\
u_o f_{o\eta} &= 1
\end{aligned}\right\}\qquad 0 < \eta < 1
$$

$$
\left.\begin{aligned}
U_{o\sigma} &= F_{o\sigma}P_{o\eta}-F_{o\eta}P_{o\sigma} \\
0 &= \ell F_{o\eta}+P_{o\eta} \\
U_o F_{o\eta} &= 1
\end{aligned}\right\}\qquad 1 < \eta < 1+r
$$

and since the vorticity is zero

$$
f_{o\eta\eta} = 0 \;; \qquad F_{o\eta\eta} = 0 \,.
$$

If we assume $u_o(-\infty,\eta) = U_o(-\infty,\eta) = 1$, the solution of this set is

$$
(4.9) \qquad
\begin{aligned}
u_o &= 1 & \qquad U_o &= 1 \\
f_o &= \eta & \qquad F_o &= \eta \\
p_o &= \lambda(1+\rho r-\eta) & \qquad P_o &= \lambda(1+r-\eta) \ .
\end{aligned}
$$

The zeroth approximation represents a parallel flow with velocity c in both layers.

The equations for the first order approximation are

$$
(4.10) \qquad
\left.
\begin{aligned}
u_{1\sigma} &= -\lambda f_{1\sigma}-p_{1\sigma} \\
0 &= 1-\lambda f_{1\eta}-p_{1\eta} \\
0 &= f_{1\eta}+u_1
\end{aligned}
\right\} \qquad 0 < \eta < 1
$$

$$
\left.
\begin{aligned}
U_{1\sigma} &= -\lambda F_{1\sigma}-P_{1\sigma} \\
0 &= 1-\lambda F_{1\eta}-P_{1\eta} \\
0 &= F_{1\eta}+U_1
\end{aligned}
\right\} \qquad 1 < \eta < 1+r
$$

subject to the vorticity conditions

$$
f_{1\eta\eta} = 0 \ ; \qquad F_{1\eta\eta} = 0 \ .
$$

The solution of this set of linear equations, subject to the appropriate boundary conditions from (4.6) is

$$
(4.11) \qquad
\begin{aligned}
f_1 &= a_1(\sigma)\eta & \qquad F_1 &= A_1(\sigma)(\eta-1)+a_1(\sigma) \\
u_1 &= -a_1(\sigma) & \qquad U_1 &= -A_1(\sigma) \\
p_1 &= -\lambda a_1(\sigma) + a_1(\sigma) & \qquad P_1 &= -\lambda A_1(\sigma)(\eta-1)-\lambda a_1(\sigma) \\
&\quad +(\eta-1-\rho r)+k_1 & &\quad +A_1(\sigma)+(\eta-1-r)+K_1
\end{aligned}
$$

where k_1 and K_1 are constants; and, due to the boundary conditions referring to pressure at $\eta = 1$ and $\eta = 1+r$, $a_1(\sigma)$ and $A_1(\sigma)$ must satisfy

$$(4.12)\qquad \begin{aligned} (1-\ell+\rho\ell)a_1(\sigma)-\rho A_1(\sigma) &= \rho K_1-k_1 \\ \ell a_1(\sigma)+(\ell r-1)A_1(\sigma) &= K_1 \end{aligned}$$

or

$$(4.13)\qquad \begin{aligned} (1-\ell+\rho\ell)a_1'(\sigma)-\rho A_1'(\sigma) &= 0 \\ \ell a_1'(\sigma)+(\ell r-1)A_1'(\sigma) &= 0 \ . \end{aligned}$$

It is here that the bifurcation phenomenon appears. If each of $a_1'(\sigma)$ and $A_1'(\sigma)$ is to be non-zero then ℓ must satisfy

$$(4.14)\qquad (1-\ell+\rho\ell)(\ell r-1)+\rho\ell = 0$$

or

$$(4.15)\qquad r(1-\rho)\ell^2-(1+r)\ell+1 = 0$$

which gives

$$\ell = \frac{1+r\pm\sqrt{(1-r)^2+4r\rho}}{2r(1-\rho)} \ .$$

These values of ℓ are just equal to the values $\lambda_1 = gh/c_1^2$; $\lambda_2 = gh/c_2^2$ which determine the critical speeds of the linear theory, as found in the preceding section. In order to obtain motions other than parallel flow we assume, of course, that ℓ satisfies (4.14).

The functions $a_1(\sigma)$ and $A_1(\sigma)$ cannot be determined at this stage; to find them we must pass to the set of equations obtained by carrying the development to next order. The equations for the second order approximation are

$$(4.16)\qquad \left.\begin{aligned} u_{2\sigma} &= a_1'(\sigma)\eta-a_1(\sigma)a_1'(\sigma)-\ell f_{2\sigma}-p_{2\sigma} \\ a_1''(\sigma)\eta &= a_1(\sigma)-\ell f_{2\eta}-p_{2\eta} \\ 0 &= f_{2\eta}+u_2-a_1^2(\sigma) \end{aligned}\right\} \qquad 0<\eta<1$$

$$
\left.\begin{aligned}
U_{2\sigma} &= A_1'(\sigma)(\eta-1)+a_1'(\sigma)-A_1(\sigma)A_1'(\sigma)-\ell F_{2\sigma}-P_{2\sigma} \\
A_1''(\sigma)(\eta-1)+a_1''(\sigma) &= A_1(\sigma)-\ell F_{2\eta}-P_{2\eta} \\
0 &= F_{2\eta}+U_2-A_1^2(\sigma)
\end{aligned}\right\} \quad 1<\eta<1+r \tag{4.16}
$$

subject to the vorticity conditions

$$
f_{2\eta\eta} = -f_{1\sigma\sigma}; \qquad F_{2\eta\eta} = -F_{1\sigma\sigma}
$$

and the boundary conditions from (4.6). The solution of this linear system is

$$
\begin{aligned}
f_2 &= a_2(\sigma)\eta-a_1''(\sigma)\ \frac{\eta^3}{6} \\
u_2 &= a_1^2(\sigma)-a_2(\sigma)+a_1''(\sigma)\ \frac{\eta^2}{2} \\
p_{2\sigma} &= a_2'(\sigma)-3a_1(\sigma)a_1'(\sigma)+[a_1'(\sigma)-\ell a_2'(\sigma)]\eta \\
&\quad - a_1''(\sigma)\ \frac{\eta^2}{2} + \ell a_1'''(\sigma)\ \frac{\eta^3}{6}
\end{aligned}
$$

(4.17)

$$
\begin{aligned}
F_2 &= -A_1''(\sigma)\frac{(\eta-1)^3}{6} -a_1''(\sigma)\frac{(\eta-1)^2}{2} +A_2(\sigma)(\eta-1)+a_2(\sigma)- \frac{a_1''(\sigma)}{6} \\
U_2 &= A_1^2(\sigma)-A_2(\sigma)+a_1''(\sigma)(\eta-1)+A_1''(\sigma)\ \frac{(\eta-1)^2}{2} \\
P_{2\sigma} &= A_2'(\sigma)-\ell a_2'(\sigma)-3A_1(\sigma)A_1'(\sigma)+a_1'(\sigma)+ \frac{\ell}{6}\ a_1'''(\sigma) \\
&\quad + [A_1'(\sigma)-\ell A_2'(\sigma)-a_1'''(\sigma)](\eta-1) \\
&\quad + [\ell a_1'''(\sigma)-A_1'''(\sigma)]\ \frac{(\eta-1)^2}{2} +\ell A_1'''(\sigma)\ \frac{(\eta-1)^3}{6}
\end{aligned}
$$

provided that the functions $a_2(\sigma)$, $A_2(\sigma)$ satisfy the ordinary differential equations

$$
\begin{aligned}
(1-\ell+\rho\ell)a_2'(\sigma)-\rho A_2'(\sigma) &= I \\
\ell a_2'(\sigma)+(\ell r-1)A_2'(\sigma) &= J\ .
\end{aligned} \tag{4.18}
$$

The equations (4.18) come, of course, from the boundary conditions on the pressure at the free surface and at the interface. In these equations the I and J are given by

$$I = \tfrac{1}{6}(3-\lambda+\rho\lambda)a_1'''(\sigma)+(\rho-1)a_1'(\sigma)$$
$$+\, 3a_1(\sigma)a_1'(\sigma)-3\rho A_1(\sigma)A_1'(\sigma)$$

and

$$J = \tfrac{1}{6}(\lambda-6r+3\lambda r^2)a_1'''(\sigma)+\tfrac{1}{6}(\lambda r^3-3r^2)A_1'''(\sigma)$$
$$+\, a_1'(\sigma)+rA_1'(\sigma)-3A_1(\sigma)A_1'(\sigma) \quad .$$

Thus I and J are defined by the first order terms. We observe also that $A_1'(\sigma)$, for example, is determined from (4.13) as soon as $a_1'(\sigma)$ is known. Since the determinant of (4.18) is zero, as we know, we have

$$(4.19) \qquad (\lambda r-1)I+\rho J = 0$$

as an equation for the determination of $a_1(\sigma)$ and $A_1(\sigma)$. If we use the relations (4.12) and (4.14) the equation (4.14) reduces to

$$(4.20) \qquad m_o a_1'''(\sigma) = m_1 a_1(\sigma)a_1'(\sigma)+m_2 a_1'(\sigma)$$

where

$$m_o = \tfrac{1}{3}\,[r(r+1)\lambda-(r^2+1+3\rho r)]$$
$$m_1 = \frac{-3}{(\lambda r-1)}\,[\lambda^2(r^2-1)+\lambda(1-2r)+1]$$
$$m_2 = -[2\lambda r(\rho-1)+r+1]+\frac{3(1-\lambda)}{(\lambda r-1)}\,K_1 \quad .$$

The general solution of (4.20) is expressible in terms of the Jacobian elliptic functions usually denoted by cn and sn ; and these lead to cnoidal waves. For these waves the quantities h and H must be interpreted as mean depths of the layers and c must be interpreted as a mean velocity. However, in what follows we will

confine our attention to the solitary wave. For the solitary wave special boundary conditions must be imposed. Since the medium is supposed to be at rest at infinity the horizontal velocity there with respect to the moving coordinate system is c which means that for the dimensionless velocities, $u(-\infty, \eta) = 1$ and $U(-\infty, \eta) = 1$. The velocities u_o and U_o satisfy these conditions and hence we can take $u_1(-\infty, \eta) = U_1(-\infty, \eta) = 0$ which means $a_1(-\infty) = A_1(-\infty) = 0$. From (4.12) we see that the horizontal velocity conditions imply

$$k_1 = 0 \; ; \qquad K_1 = 0 \; .$$

The vertical velocity in the lower layer is given by (4.7) and using what we have found for f_o and f_1

$$v = \varepsilon\sqrt{\varepsilon}\; a_1'(\sigma)\eta + \ldots \quad .$$

The condition that the vertical velocity is zero at $-\infty$ is satisfied in the lowest order approximation to v by taking $a_1'(-\infty) = 0$. Since the vertical velocity at the crest of the wave is zero we take $a_1'(0) = 0$. In addition to the above conditions on $a_1(\sigma)$ we satisfy the condition $u_2(-\infty, \eta) = 0$ by taking $a_2(-\infty) = a_1''(-\infty) = 0$ as we can see from (4.17). Subject to these conditions, the integration of (4.20) gives

$$m_o a_1''(\sigma) = \frac{m_1 a_1^2(\sigma)}{2} + m_2 a_1(\sigma) \; . \tag{4.21}$$

If we multiply (4.21) by $a_1'(\sigma)$ and integrate we find

$$m_o [a_1'(\sigma)]^2 = \frac{m_1}{3}\, a_1^3(\sigma) + m_2 a_1^2(\sigma)$$

and from this

$$a_1(\sigma) = \frac{-3m_2}{m_1}\,\mathrm{sech}^2\,\frac{\sigma}{2}\sqrt{\frac{m_2}{m_o}}\ .$$

The function $A_1(\sigma)$ is then given by

$$A_1(\sigma) = \frac{\ell a_1(\sigma)}{1-\ell r}\quad .$$

If we now suppose that what we have found gives a sufficiently accurate representation of the solitary wave; if we use the symbol $\cong$ to denote approximation; and if we return to the x,γ variables, a collection of the results gives the following:

$$\bar{f}(x,\gamma) \cong h\eta - \frac{3m_2}{m_1}(\ell-\lambda)h\eta\,\mathrm{sech}^2\,\frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)}$$

$$\bar{u}(x,\gamma) \cong c + \frac{3m_2}{m_1}(\ell-\lambda)c\,\mathrm{sech}^2\,\frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)}$$

$$\bar{p}(x,\gamma) \cong \delta c^2[\lambda(1+\rho r-\eta) - \frac{3m_2}{m_1}(\ell-\lambda)(1-\ell\eta)\,\mathrm{sech}^2\,\frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)}]$$

$$\bar{v}(x,\gamma) \cong c(\ell-\lambda)^{\frac{3}{2}}\,\frac{3m_2}{m_1}\sqrt{\frac{m_2}{m_o}}\,\eta\,\mathrm{sech}^2\,\frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)}\cdot\tanh\,\frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)}$$

$$\bar{F}(x,\gamma) \cong h\eta - \frac{3m_2(\ell-\lambda)h}{m_1(1-\ell r)}[\ell(\eta-1)+1-\ell r]\,\mathrm{sech}^2\,\frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)}$$

$$\bar{U}(x,\gamma) \cong c + \frac{3m_2}{m_1}\frac{(\ell-\lambda)}{(1-\ell r)}\ell c\,\mathrm{sech}^2\,\frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)}$$

$$\bar{P}(x,\gamma) \cong \rho\delta c^2[\lambda(1+r-\eta) - \frac{3m_2}{m_1}\frac{(\ell-\lambda)\ell^2(1+r-\eta)}{(1-\ell r)}\,\mathrm{sech}^2\,\frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)}$$

In these results

$$\eta = \frac{\gamma}{ch} \qquad \lambda = \frac{gh}{c^2} \qquad \rho = \frac{\Delta}{\delta} \qquad r = \frac{H}{h}$$

where c is the velocity of the solitary wave; δ and Δ are the respective densities of the lower and upper layers; h and H are the

respective depths of the lower and upper layers at infinity. In terms of the vertical distance, y_o, from the bottom at $x = -\infty$,

$$\gamma = \int_0^{y_o} \psi_y(-\infty, y)dy = \int_0^{y_o} c\,dy = cy_o \ .$$

The constants m_i are given by

$$m_o = \frac{1}{3}\,[r(r+1)\ell-(r^2+1+3\rho r)]$$
$$m_1 = \frac{-3}{(\ell r-1)}\,[\ell^2(r^2-1)+\ell(1-2r)+1]$$
$$m_2 = -\,[2\ell r(\rho-1)+1+r]$$

and ℓ is a zero of

$$r(1-\rho)\ell^2-(1+r)\ell+1 = 0$$

that is

$$\ell = \ell_1 = \frac{(1+r)+\sqrt{(1-r)^2+4r\rho}}{2r(1-\rho)}$$

or

$$\ell = \ell_2 = \frac{1+r-\sqrt{(1-r)^2+4r\rho}}{2r(1-\rho)} \ .$$

It will be noticed that there must be a restriction on $\ell-\lambda$ since its sign must be such that $\frac{m_2(\ell-\lambda)}{m_o} > 0$. This quantity appears under a square root sign in various places above. It can be shown, however, that m_2/m_o is positive so that the condition becomes $\ell-\lambda > 0$.

5. Discussion of the Results

According to the approximate expression for $\overline{f}(x,\gamma)$, the maximum elevation or depression of the solitary wave at the interface of the two layers is

$$q \cong |\bar{f}(0,ch)-h|$$

and from this

$$(5.1) \qquad \pm q \cong -\frac{3m_2}{m_1}(\ell-\lambda)h \; .$$

In terms of this amplitude the velocity of the wave is given by

$$\frac{gh}{c^2} = \lambda \cong \ell \pm \frac{q}{h}\,\frac{m_1}{3m_2}$$

or

$$(5.2) \qquad \frac{gh}{c^2} \cong \ell \pm q\,\frac{[\ell^2(r^2-1)+\ell(1-2r)+1]}{h(\ell r-1)[2\ell r(\rho-1)+1+r]} \; .$$

The stream line functions can be expressed in the form

$$(5.3) \qquad \begin{aligned} \bar{f}(x,\gamma) &\cong h\eta \pm q\eta \operatorname{sech}^2 \frac{x}{2h}\sqrt{\pm q\left(\frac{m_1}{-3m_0 h}\right)} \\ \bar{F}(x,\gamma) &= h\eta \pm q\,\frac{[\ell(\eta-1-r)+1]}{1-\ell r}\operatorname{sech}^2 \frac{x}{2h}\sqrt{\pm q\left(\frac{m_1}{-3m_0 h}\right)} \end{aligned}$$

where

$$\frac{m_1}{-3m_0} = \frac{3[\ell^2(r^2-1)+\ell(1-2r)+1]}{(\ell r-1)[r(r+1)\ell-(r^2+1+3\rho r)]}$$

and the sign associated with q is to be chosen so that

$$\pm q\left(\frac{m_1}{-3m_0}\right) > 0 \; .$$

It can be shown that

$$(\ell r-1)[r(r+1)\ell-(r^2+1+3\rho r)] > 0 \; .$$

Hence the sign to be taken is

$$+ \text{ if } \lambda^2(r^2-1)+\lambda(1-2r)+1 > 0$$

and

$$- \text{ if } \lambda^2(r^2-1)+\lambda(1-2r)+1 < 0 \ .$$

The possibility of having $\lambda^2(r^2-1)+\lambda(1-2r)+1 = 0$ will be considered later.

The solitary wave for a single layer of depth h and density equal to the constant δ can be obtained from the above by setting $r = 0$. This gives

$$\lambda = 1 \ , \qquad \frac{m_1}{-3m_o} = 3 > 0$$

so that

$$\bar{f}(x,\gamma) \cong h\eta + q\eta \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3q}{h}}$$

$$\bar{u}(x,\gamma) \cong c - q\frac{c}{h} \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3q}{h}}$$

$$\bar{p}(x,\gamma) \cong \delta gh(1-\eta) + \delta c^2 \frac{q}{h}(1-\eta) \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3q}{h}}$$

$$\bar{v}(x,\gamma) \cong -c\sqrt{\frac{3q}{h}}\,\frac{q\eta}{h} \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3q}{h}} \cdot \tanh \frac{x}{2h}\sqrt{\frac{3q}{h}}$$

and the velocity of the wave from (5.1) is

$$\frac{gh}{c^2} \cong 1 - \frac{q}{h}$$

or

$$\frac{c^2}{gh} \cong 1 + \frac{q}{h}$$

which is the Scott Russell formula. The stream lines are given by

$$y = \bar{f}(x,\gamma) \cong y_s\eta = y_s\frac{\gamma}{ch} = \frac{y_o}{h}\,y_s$$

where

$$y_s \cong h + q \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3q}{h}} \tag{5.4}$$

is the equation of the profile of the solitary wave at the free surface. In terms of y_s the horizontal velocity component is

$$\tilde{u}(x,y) = \bar{u}(x,\gamma) \cong \frac{c}{2}\,(2h-y_s) \tag{5.5}$$

and the pressure is

$$\tilde{p}(x,y) = \bar{p}(x,\gamma)$$

$$\cong \delta g[h(1-\eta)+(1+\frac{q}{h})q(1-\gamma) \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3q}{h}}\,]$$

$$\tilde{p}(x,y) \cong \delta g(y_s-y) \tag{5.6}$$

which shows that the hydrostatic law holds up to second order terms in q. The approximation to the vertical velocity component is of higher order than the approximations to the other quantities. It is

$$\begin{aligned}\tilde{v}(x,y) &= \bar{v}(x,y)\\ &= -c\sqrt{\frac{3q}{h}}\;\frac{qy_o}{h^2}\;(y_s-h)\tanh \frac{x}{2h}\sqrt{\frac{3q}{h}}\,.\end{aligned} \tag{5.7}$$

The results (5.4), (5.5), (5.6) and (5.7) coincide with those given by Keller [15].

For the case of the two-layer medium, ℓ may take either the value

$$\ell = \ell_2 = \frac{1+r-\sqrt{(1-r)^2+4\rho r}}{2r(1-\rho)} = \frac{2}{1+r+\sqrt{(1-r)^2+4\rho r}}$$

which is positive, or the value

$$\ell = \ell_1 = \frac{1+r+\sqrt{(1-r)^2+4\rho r}}{2r(1-\rho)} = \frac{2}{1+r-\sqrt{(1-r)^2+4\rho r}}$$

which is also positive because $\rho < 1$. These values lead to two very different kinds of solitary wave behavior.

Type 1. If $\lambda = \lambda_2$, then

$$\frac{m_1}{-3m_0} = \frac{3[(\lambda_2 r-1)^2+\lambda_2(1-\lambda_2)]}{(1-\lambda_2 r)[3\rho r+(1-r\lambda_2)+r^2(1-\lambda_2)]}$$

and since

$$1-\lambda_2 r = \frac{1-r+\sqrt{(1-r)^2+4\rho r}}{1+r+\sqrt{(1-r)^2+4\rho r}} > 0$$

$$1-\lambda_2 = \frac{-(1-r)+\sqrt{(1-r)^2+4\rho r}}{1+r+\sqrt{(1-r)^2+4\rho r}} > 0$$

$\frac{m_1}{-3m_0}$ is positive and the sign to be associated with q in (5.1), (5.2) and (5.3) is the positive sign. If we introduce

$$d_1 = \frac{(\lambda_2 r-1)^2 + \lambda_2(1-\lambda_2)}{3\rho r+(1-r\lambda_2)+r^2(1-\lambda_2)} > 0$$

we see from (5.3) that the stream lines are given by

$$y \cong h\eta+q\eta \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3qd_1}{h(1-\lambda_2 r)}} \qquad 0 < \eta \leq 1$$

and

$$y \cong h\eta+q\,\frac{[\lambda_2(\eta-1-r)+1]}{1-\lambda_2 r}\operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3qd_1}{h(1-\lambda_2 r)}} \qquad 1 \leq \eta < 1+r$$

and we see that these are lines of <u>elevation</u> above the stream lines of the uniform flow. The wave profile at the free surface is given by

$$y_s \cong h(1+r)+\frac{q}{1-\lambda_2 r}\operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3qd_1}{h(1-\lambda_2 r)}}$$

or

$$y_s \cong h(1+r)+Q\,\text{sech}^2\,\frac{x}{2h}\sqrt{\frac{\beta Q d_1}{h}}$$

where

$$Q = \frac{2}{1-\lambda_2 r}$$

is the amplitude of the wave at the free surface. The wave profile at the interface is given by

$$y_i \cong h+(1-\lambda_2 r)Q\,\text{sech}^2\,\frac{x}{2h}\sqrt{\frac{3Qd_1}{h}}\ .$$

The amplitude of the wave at the free surface is $1/(1-\lambda_2 r)$ times the amplitude at the interface. If the densities of the layers are nearly equal this factor

$$\frac{1}{1-\lambda_2 r} = \frac{1+r+\sqrt{(1-r)^2+4\rho r}}{1-r+\sqrt{(1-r)^2+4\rho r}}$$

is approximately equal to 1+r. If the depths of the layers are the same, the amplitude at the free surface is about twice the amplitude of the wave at the interface.

The wave velocity when $\lambda = \lambda_2$ is given by

$$\frac{gh}{c^2} \cong \lambda_2 + \frac{Q}{h}\,\frac{(\lambda_2 r-1)^2+\lambda_2(1-\lambda_2)}{[2\lambda_2 r(1-\rho)-(1+r)]}\ .$$

For the case of layers of equal depth

$$r = 1 \qquad\qquad \lambda_2 = \frac{1}{1+\sqrt{\rho}}$$

and the wave velocity is given by

$$\frac{gh}{c^2} \cong \frac{1}{1+\sqrt{\rho}} - \frac{Q}{2h} \cdot \frac{1}{1+\sqrt{\rho}}$$

$$\frac{c^2}{gh} \cong (1+\sqrt{\rho})\ [1 + \frac{Q}{2h}]\ .$$

This coincides with the Scott Russell formula for $\rho = 1$.

Type 2. If $\ell = \ell_1$ the sign of

$$\ell_1^2(r^2-1)+\ell_1(1-2r)+1 = (\ell_1 r-1)^2-\ell_1(\ell_1-1)$$
$$= \frac{\ell_1^2}{2}\ [r^2-r+2\rho r+(r-2)\sqrt{(1-r)^2+4\rho r}]$$

may be positive or negative. It is negative for example if $0 \leq r < 1$; and it is positive if $2 \leq r$. In either case the amplitude of the wave at the free surface is $1/|1-\ell_1 r|$ times the amplitude of the wave at the interface. This factor has the value

$$0 < \frac{1}{\ell_1 r-1} = \frac{r+1-\sqrt{(1-r)^2+4\rho r}}{r-1+\sqrt{(1-r)^2+4\rho r}} < \frac{4r(1-\rho)}{(1+r)[r-1+\sqrt{(1-r)^2+4\rho r}]}$$

and we see that if the densities of the layers are nearly equal, the factor is small. If $r = 1$ the factor has the value

$$\frac{1}{\ell_1 r-1} = \frac{1-\sqrt{\rho}}{\sqrt{\rho}}\ .$$

The waves of the present type tend to have quite small amplitudes at the free surface. They are internal solitary waves, and, as we will see in a moment, the wave at the interface may be a wave either of <u>elevation</u> above the undisturbed line or of <u>depression</u> below it. In connection with the above remark about the amplitude at the free

surface it should be noted that there is a stream line in the upper layer which is parallel to the bottom. As we can see from (5.3) the horizontal stream line occurs when

$$\eta = 1+r - \frac{1}{\lambda_1}$$

$$= \frac{1+r+\sqrt{(1+r)^2-4(1-\rho)r}}{2}$$

When the density difference is small this line is near the free surface.

If $\lambda_1^2(r^2-1)+\lambda_1(1-2r)+1$ is positive the positive sign must be taken in (5.3) and the stream lines are

$$y \cong h\eta+q\eta \operatorname{sech}^2 \frac{x}{2h}\sqrt{q\left(\frac{m_1}{-3m_0h}\right)} \qquad 0 < \eta < 1$$

and

$$y \cong h\eta-q\,\frac{[\lambda_1(\eta-1-r)+1]}{\lambda_1 r-1} \operatorname{sech}^2 \frac{x}{2h}\sqrt{q\left(\frac{m_1}{-3m_0h}\right)} \qquad 1 < \eta < 1+r.$$

These are lines of <u>elevation</u> for $0 < \eta < 1+r-\frac{1}{\lambda_1}$. For $1+r-\frac{1}{\lambda_1} < \eta < 1+r$ the stream lines in the upper layer are lines of <u>depression</u>. The internal wave of elevation at the interface is

$$y_i \cong h+q \operatorname{sech}^2 \frac{x}{2h}\sqrt{q\left(\frac{m_1}{-3m_0h}\right)}$$

and the small amplitude wave of depression at the free surface is

$$y_s \cong h(1+r) - \frac{q}{\lambda_1 r-1} \operatorname{sech}^2 \frac{x}{2h}\sqrt{q\left(\frac{m_1}{-3m_0h}\right)} .$$

If $\ell_1^2(r^2-1)+\ell_1(1-2r)+1$ is negative the negative sign must be taken in (5.3) and the stream lines

$$y \cong h\eta - q\eta \operatorname{sech}^2 \frac{x}{2h}\sqrt{q\left(\frac{m_1}{+3m_o h}\right)} \qquad 0<\eta<1$$

and

$$y \cong h\eta + q\,\frac{[\ell_1(\eta-1-r)+1]}{\ell_1 r-1} \operatorname{sech}^2 \frac{x}{2h}\sqrt{q\left(\frac{m_1}{+3m_o h}\right)} \qquad 1<\eta<1+r.$$

These are lines of <u>depression</u> for $0<\eta<1+r-\frac{1}{\ell_1}$. In the remaining part of the upper layer the stream lines are lines of <u>elevation</u>. The internal solitary wave of depression at the interface is

$$y_i = h-q \operatorname{sech}^2 \frac{x}{2h}\sqrt{q\left(\frac{m_1}{3m_o h}\right)}$$

and the small amplitude wave of elevation at the free surface is

$$y_s = h(1+r)+q\left(\frac{1}{\ell_1 r-1}\right) \operatorname{sech}^2 \frac{x}{2h}\sqrt{q\left(\frac{m_1}{3m_o h}\right)} .$$

For a medium in which the depths of the layers are the same

$$r = 1 \; ; \qquad \ell_1 = \frac{1}{1-\sqrt{\rho}}$$

and

$$\ell_1^2(r^2-1)+\ell_1(1-2r)+1 = 1-\ell_1 = \frac{-\sqrt{\rho}}{1-\sqrt{\rho}} < 0 .$$

Therefore in this case the internal solitary wave at the interface is a depression wave.

The wave speed for waves of Type 2 is

$$\frac{gh}{c^2} \cong \ell_1 \pm q \frac{[\ell_1^2(r^2-1)+\ell_1(1-2r)+1]}{h(\ell_1 r-1)[2\ell_1 r(\rho-1)+1+r]} \ .$$

Since $\ell_1 r-1 > 0$ and $2\ell_1 r(\rho-1)+1+r < 0$, the formula for the speed can be written

$$\frac{gh}{c^2} \simeq \ell_1 - q \frac{|\ell_1^2(r^2-1)+\ell_1(1-2r)+1|}{h(\ell_1 r-1)[2\ell_1 r(1-\rho)-(1+r)]} \ .$$

For $r = 1$ this is

$$\frac{gh}{c^2} \cong \frac{1}{1-\sqrt{\rho}} - \frac{q}{2h\sqrt{\rho}}$$

$$\cong \frac{1}{1-\sqrt{\rho}} \left[1- \frac{q(1-\sqrt{\rho})}{2h\sqrt{\rho}}\right]$$

or to the same order of approximation

$$\frac{c^2}{gh} \cong (1-\sqrt{\rho}) \left[1+ \frac{g}{2h} \left(\frac{1-\sqrt{\rho}}{\sqrt{\rho}}\right)\right] \ .$$

Let us suppose now that

$$\ell = \ell_1 = \frac{1+r+\sqrt{(1-r)^2+4\rho r}}{2r(1-\rho)} = \frac{2}{1+r-\sqrt{(1-r)^2+4\rho r}}$$

and that the depth ratio r and the density ratio ρ are such that

$$\ell_1^2(r^2-1)+\ell_1(1-2r)+1 = \frac{\ell_1^2}{2} \left[r^2-r+2\rho r+(r-2)\sqrt{(1-r)^2+4\rho r}\ \right]$$

vanishes. In order to discuss this situation, for which m_1 is zero, we need to return to the differential equation which determines $a_1(\sigma)$. If $m_1 = 0$, the equation is

$$m_o[a_1'(\sigma)]^2 = m_2 a_1^2(\sigma) \; .$$

The general solution of this equation, subject to our conditions at infinity, is

$$a_1(\sigma) = \begin{cases} \alpha e^{-\sigma\sqrt{\frac{m_2}{m_o}}} & \sigma > 0 \\ \beta e^{\sigma\sqrt{\frac{m_2}{m_o}}} & \sigma < 0 \; . \end{cases}$$

This solution is continuous with a continuous first derivative if and only if each of the constants α and β is zero. The choice of the latter value for α and β leads to $f_1 = 0$ and we are left with a parallel flow.

It might be supposed that the remaining solutions may lead to something more interesting. For example, the discontinuous solution

$$a_1(\sigma) = \alpha\mu(\sigma)e^{-\sigma\sqrt{\frac{m_2}{m_o}}} + \beta\mu(-\sigma)e^{\sigma\sqrt{\frac{m_2}{m_o}}}$$

(where $\alpha \neq \beta$ and $\mu(\sigma)$ is the unit function) suggests that internal bores may exist at the interface. If we apply the first of the shock conditions, which requires that mass be conserved across the discontinuity, we find (using the approximations under consideration) that β must be equal to the negative of α. Thus the first shock condition requires

$$a_1(\sigma) = \alpha(\text{signum}\,\sigma)\, e^{-|\sigma|\sqrt{\frac{m_2}{m_o}}} \; .$$

Now if we apply the second shock condition, which requires conservation of momentum across the discontinuity, it turns out that an additional relation between r and ρ must be satisfied. This relation, however, is incompatible with

$$r^2-r+2\rho r+(r-2)\sqrt{(1-r)^2+4r\rho} = 0 ,$$

and hence a bore with the properties our theory requires cannot be admitted.

The only remaining possibility is to take

$$a_1(\sigma^-) = \alpha e^{-|\sigma^-|\sqrt{\frac{m_2}{m_o}}}$$

but this gives a solution for which the vorticity is not zero.

It seems then that for the special circumstance

$$r^2-r+2\rho r+(r-2)\sqrt{(1-r)^2+4r\rho} = 0$$

the only flow of the second type which our theory allows is one of parallel flow.

Part II

Liquids with a Continuous Variation in Density

1. Introduction

This part presents an analysis of the problem of solitary waves in an incompressible medium with a continuous variation of density. The general theory is developed under the assumption that the density variation is arbitrary, but in solving concrete problems we assume that the density of the medium in the equilibrium state increases exponentially with the depth of the medium in the channel. The method we use is applicable to other kinds of variation of density, but these in general would lead to the necessity of solving certain ordinary second order linear differential equations with variable coefficients instead of those with constant coefficients which arise for the case of exponential variation of density.

Although the analysis in this part is similar to the analysis of Part I for solitary waves in a medium having two layers, there is a minimum of reference to Part I and hence the development of this part is virtually self contained. The general nonlinear problem is formulated in Section 2. In Section 3 the nonlinear problem is linearized and it is shown that the linear theory furnishes an infinite number of critical speeds.

In Section 4 there is a return to the nonlinear problem and it is analyzed by using a variant of the shallow water technique invented by Friedrichs. The analysis shows that solitary wave solutions (and cnoidal wave solutions) appear as bifurcations from states of uniform

parallel flow with speeds coinciding with the critical speeds found in Section 3. The solitary wave solutions can be denoted by S_n where the subscript n indicates that the solitary wave defined by S_n travels with a speed in the neighborhood of c_n, the n^{th} critical speed of the linear theory. The critical speed c_n approaches zero as $n \rightarrow \infty$, and in fact the set $\{c_n\}$ is a discrete set of monotonically decreasing values which has zero as its only limit point. The solitary wave derived from S_1 travels with a speed in the neighborhood of the highest critical speed, and it has properties similar to those of the ordinary solitary wave in a medium of constant density. The solutions S_n $(n \neq 1)$ are found to yield internal solitary waves. The amplitude of each of these waves at the free surface is small compared with the maximum departure of certain internal stream lines from the parallel lines which they approach at infinity. The state of the medium for these waves can be described roughly by saying that, starting at the bottom, the channel is divided into adjacent horizontal strips of nearly equal widths (except the uppermost one) which alternately contain only stream lines of elevation and only stream lines of depression with respect to the undisturbed parallel flow, of course. The horizontal division lines for the strips are themselves stream lines; the highest such division line is found to lie a small distance below the free surface; and there are n strips. The maximum departure of the stream lines from the equilibrium lines of parallel flow in any one of these strips (except the one which contains the free surface) is slightly greater than the maximum departure of the stream lines in the strip immediately below. These properties and others are discussed in some detail in Section 5, where particular attention is paid to the waves derived from S_1 and S_2.

2. Formulation of the Problem for the Case of Variable Density

For the purpose of studying two-dimensional waves of permanent type in an inviscid, incompressible medium of variable density we suppose that a cross section of the medium in the equilibrium state fills an infinite horizontal strip. The medium is assumed to be supported by a rigid bottom, and the upper surface is assumed to be a free surface at which the pressure is zero and there are no geometric constraints. We also suppose that a disturbance of some kind in the medium initially at rest has created a wave of permanent type which moves to the left with constant velocity c; and that the character of this wave is observed from a coordinate system which moves with the wave. We take the x-axis of this system so that it coincides with the bottom of the channel and the y-axis so that it is positive upward and passes through a crest of the wave. With respect to these axes the wave is stationary; and the velocity of the medium at infinity is horizontal and equal to c. We take the depth of the medium at infinity to be h.

The differential equations for the flow under these conditions are well known to be

$$
(2.1)\qquad
\begin{aligned}
\tilde\delta(\tilde u\tilde u_x+\tilde v\tilde u_y) &= -\tilde p_x\ ,\\
\tilde\delta(\tilde u\tilde v_x+\tilde v\tilde v_y) &= -\tilde\delta g-\tilde p_y\ ,\\
\tilde\zeta &= \tilde v_x-\tilde u_y\ ,\\
\tilde u_x+\tilde v_y &= 0\ ,\\
\tilde u\tilde s_x+\tilde v\tilde s_y &= 0\ ,\\
\tilde u\tilde\delta_x+\tilde v\tilde\delta_y &= 0\ ,
\end{aligned}
$$

where $\tilde{\delta}(x,y)$ is the density; $\tilde{u}(x,y)$, $\tilde{v}(x,y)$ are the horizontal and vertical velocity components; $\tilde{p}(x,y)$ is the pressure; g is the acceleration due to gravity; $\tilde{\zeta}(x,y)$ is the vorticity; and $\tilde{s}(x,y) = 0$ is the equation of a boundary curve.

If the stream function is $\psi(x,y)$ so that

$$\tilde{u} = \psi_y(x,y) \; ; \qquad \tilde{v} = -\psi_x(x,y) \; ,$$

the totality of stream lines is given by

$$\psi(x,y) = \gamma$$

where γ ranges over a finite interval. The function $\psi(x,y)$ can be chosen so that for the bottom

$$\gamma = \psi(x,0) = 0$$

and then the value of γ which determines the free surface can be taken to be

$$\gamma = ch \; .$$

It is assumed that to each value of γ there corresponds one and only one stream line which is given by solving $\gamma = \psi(x,y)$ for y, namely

$$y = \bar{f}(x,\gamma) \; .$$

Let us use x and γ as independent variables. As the basic dependent variables we choose the stream line functions $\bar{f}(x,\gamma)$; the horizontal velocity component $\bar{u}(x,\gamma) = \tilde{u}[x,\bar{f}(x,\gamma)]$; and the pressure $\bar{p}(x,\gamma) = \tilde{p}[x,\bar{f}(x,\gamma)]$. With these variables

$$\bar{f}_\gamma = \frac{1}{\psi_y} = \frac{1}{\bar{u}}\ ; \qquad \bar{f}_x = \frac{-\psi_x}{\psi_y} = \frac{\bar{v}}{\bar{u}} \tag{2.2}$$

and we find, as we did in Section 2 of Part I, that the first three of equations (2.1) reduce to

$$\begin{aligned} \bar{\delta}\bar{u}_x &= -\bar{f}_\gamma \bar{p}_x + \bar{f}_x \bar{p}_\gamma \\ \bar{\delta}(\bar{u}\bar{f}_{xx} + \bar{u}_x \bar{f}_x) &= -\bar{\delta} g \bar{f}_\gamma - \bar{p}_\gamma \\ \bar{\zeta} &= \frac{\partial}{\partial x}\left(\frac{\bar{f}_x}{\bar{f}_\gamma}\right) - \frac{1}{2}\frac{\partial}{\partial \gamma}\left(\frac{1+\bar{f}_x^2}{\bar{f}_\gamma^2}\right) . \end{aligned} \tag{2.3}$$

The fourth and fifth of equations (2.1) are automatically satisfied by the new variables; and the sixth equation, which arises from the assumption of incompressibility, becomes

$$\tilde{u}\tilde{\delta}_x + \tilde{v}\tilde{\delta}_y = \bar{u}\bar{\delta}_x = 0$$

which shows that $\bar{\delta} = \tilde{\delta}[x, \bar{f}(x,\gamma)]$ is a function of γ only. With the equations (2.3) we see from (2.2) that we also have

$$\bar{u}\bar{f}_\gamma = 1 . \tag{2.4}$$

In order to make the equations dimensionless we introduce the variables

$$\xi = \frac{x}{h} \qquad v = \frac{\bar{v}}{c} \qquad \zeta = \frac{h}{c}\,\bar{\zeta}$$

$$\eta = \frac{\gamma}{ch} \qquad p = \frac{\bar{p}}{\delta_o c^2} \qquad \delta = \frac{\bar{\delta}}{\delta_o}$$

$$u = \frac{\bar{u}}{c} \qquad f = \frac{\bar{f}}{h} \qquad \lambda = \frac{gh}{c^2}$$

where δ_o is the density of the medium at the bottom of the channel. The equations in dimensionless form are

$$
(2.5)\qquad
\begin{aligned}
\delta u_\xi &= -f_\eta p_\xi + f_\xi p_\eta \\
\delta(u f_{\xi\xi} + u_\xi f_\xi) &= -\lambda \delta f_\eta - p_\eta \\
u f_\eta &= 1 \\
\zeta &= \frac{\partial}{\partial \xi}\left(\frac{f_\xi}{f_\eta}\right) - \frac{1}{2}\frac{\partial}{\partial \eta}\left(\frac{1+f_\xi^2}{f_\eta^2}\right)
\end{aligned}
$$

where δ, as we know, is a function of η only. These equations must hold for the strip

$$-\infty < \xi < \infty ; \qquad 0 < \eta < 1 .$$

The boundary condition at the bottom is

$$(2.6)\qquad f(\xi,0) = 0$$

and the boundary condition at the free surface is

$$(2.7)\qquad p(\xi,1) = 0 .$$

The equation which expresses Bernoulli's law is found by eliminating u from the first three of equations (2.5) and then integrating the resulting equation. The elimination of u gives

$$\frac{\delta}{f_\eta}\frac{\partial}{\partial \xi}\left(\frac{1}{f_\eta}\right) + \delta\frac{f_\xi}{f_\eta}\frac{\partial}{\partial \xi}\left(\frac{f_\xi}{f_\eta}\right) + \lambda\delta f_\xi + p_\xi = 0$$

and integration along η = cons't. gives

$$\frac{\delta}{2}\left(\frac{1+f_\xi^2}{f_\eta^2}\right)+\lambda\delta f+p=C(\eta)\,. \tag{2.8}$$

If we differentiate the Bernoulli equation (2.8) with respect to η and use the second and third equation of (2.5) we find

$$\frac{\delta}{2}\frac{\partial}{\partial\eta}\left(\frac{1+f_\xi^2}{f_\eta^2}\right)-\delta\frac{\partial}{\partial\xi}\left(\frac{f_\xi}{f_\eta}\right)+\delta'\left[\frac{1+f_\xi^2}{2f_\eta^2}+\lambda f\right]=C'(\eta)\,. \tag{2.9}$$

This is the equation which must be satisfied by the stream line function $f(\xi,\eta)$. The equation can also be interpreted as a relation between the vorticity, pressure and density:

$$\zeta+\frac{\delta'}{\delta^2}\,p=-\frac{\partial}{\partial\eta}\left[\frac{C(\eta)}{\delta}\right]\,.$$

3. The Linearized Equations. Critical Speeds

The equations (2.5) and the boundary conditions (2.6) and (2.7) are satisfied by

$$u=1$$

$$f=\eta$$

$$p=\lambda\int_\eta^1\delta(\eta)d\eta$$

which expresses a state of parallel flow with constant velocity. Let us consider a wave motion which is a small disturbance superposed on this state of constant flow. Thus we write

$$u=1+u^*\,,$$

$$f=\eta+f^*\,,$$

$$p=\lambda\int_\eta^1\delta(\eta)d\eta+p^*\,,$$

with u^*, f^*, p^* regarded as small quantities. Then, if we substitute these quantities in (2.5), (2.6) and (2.7) and assume that the disturbance is so small that second order products involving the starred quantities can be neglected, we find the linear equations

$$\delta u^*_\xi = -p^*_\xi - \lambda \delta f^*_\xi$$
$$\delta f^*_{\xi\xi} = -\lambda \delta f^*_\eta - p^*_\eta$$
$$u^* + f^*_\eta = 0$$

subject to the boundary conditions

$$f^*(\xi,0) = 0$$
$$p^*(\xi,1) = 0 \quad .$$

From these equations we find that $f(\xi,\eta)$ must satisfy

$$\delta(f^*_{\xi\xi\xi} + f^*_{\xi\eta\eta}) + \delta'(f^*_{\xi\eta} - \lambda f^*_\xi) = 0 \tag{3.1}$$

subject to the boundary conditions

$$f^*_\xi(\xi,0) = 0 \ , \tag{3.2}$$
$$f^*_{\xi\eta}(\xi,1) = \lambda f^*_\xi(\xi,1) \ . \tag{3.3}$$

From here on we assume that the density decreases exponentially as we go upward from the bottom of the channel to the top. In fact, we assume

$$\delta(\eta) = e^{-2k\eta} .$$

The equation (3.1) then becomes

$$f^*_{\xi\xi\xi}+f^*_{\xi\eta\eta}-2k(f^*_{\xi\eta}-\lambda f^*_{\xi}) = 0 . \tag{3.4}$$

This equation and the boundary condition (3.2) are satisfied by

$$f_{\xi} = A \cos \nu(\xi+B)e^{k\eta} \sin \eta \sqrt{2k\lambda-k^2-\nu^2}$$

where A, B and ν are arbitrary constants except that A must be small. In order to satisfy the boundary condition (3.3) we must have

$$(k-\lambda) \tan\sqrt{2k\lambda-k^2-\nu^2} + \sqrt{2k\lambda-k^2-\nu^2} = 0 . \tag{3.5}$$

If we set

$$\alpha_{\nu} = \sqrt{2k\lambda-k^2-\nu^2} \tag{3.6}$$

(3.5) can be written

$$\tan \alpha_{\nu} = \frac{2k\alpha_{\nu}}{\alpha_{\nu}^2+\nu^2-k^2} . \tag{3.7}$$

This equation relates the wave number ν to the speed of the wave and the exponential variation of density. If the wave number ν (or the period $T = 2\pi/\nu$) is given for a fixed depth and fixed k, the equation (3.7) has an infinite number of roots α_{ν}, and since

$$2k\lambda = 2k\,\frac{gh}{c^2} = \alpha_\nu^2+\nu^2+k^2 \quad ,$$

there are an infinite number of speeds corresponding to a prescribed wave length.

In Part I we found that the critical speeds could be defined in various ways. Here we determine them as the speeds corresponding to waves whose wave length tends to infinity. For the case of exponential variation of density we easily see from (3.7) and (3.6) that these speeds are given by the roots of

$$\tan\alpha = \frac{2k\alpha}{\alpha^2-k^2} \tag{3.8}$$

where

$$\alpha = \sqrt{2k\lambda_n-k^2}$$

and

$$\lambda_n = \frac{gh}{c_n^2} = \frac{\alpha^2+k^2}{2k} \quad .$$

The equation (3.8) has an infinite number of real positive roots, and no pure imaginary root of magnitude greater than zero. It furnishes, in other words, a discrete infinite spectrum of critical speeds. The equation (3.8) will be discussed in greater detail in the next section. For the time being, it suffices to remark that our experience with a medium of two layers leads us to suspect that there exist solitary and cnoidal waves with speeds in the neighborhoods of the critical speeds given by (3.8).

4. Solitary Waves Near the Critical Speeds

We return now to the nonlinear equation (2.5). For a variation of density in the form of an exponential decrease going upward:

$$\delta = e^{-2k\eta}, \qquad\qquad k > 0 ,$$

these equations are

$$\begin{aligned} e^{-2k\eta}u_\xi &= -f_\eta p_\xi + f_\xi p_\eta \\ e^{-2k\eta}(uf_{\xi\xi}+u_\xi f_\xi) &= -\lambda e^{-2k\eta}f_\eta - p_\eta \\ uf_\eta &= 1 \end{aligned} \tag{4.1}$$

subject to the boundary conditions

$$\begin{aligned} f(\xi,0) &= 0 \\ p(\xi,1) &= 0 \quad . \end{aligned} \tag{4.2}$$

We proceed to investigate a solitary wave which moves with a speed such that $\lambda = gh/c^2$ is near some value ℓ, which is to be determined, but which will be found to have one of the values fixed by giving c any of the critical values just described above. We use the same general method that was used in Part I. That is, we first write the equations (4.1) in the form

$$\begin{aligned} e^{-2k\eta}u_\xi &= -f_\eta p_\xi + f_\xi p_\eta \\ e^{-2k\eta}(uf_{\xi\xi}+u_\xi f_\xi) &= (\ell-\lambda)\ e^{-2k\eta}f_\eta - \ell e^{-2k\eta}f_\eta - p_\eta \\ uf_\eta &= 1\ . \end{aligned} \tag{4.3}$$

Then we set

$$\ell - \lambda = \varepsilon$$

and introduce the new independent variable

$$\sigma = \xi\sqrt{\varepsilon} \quad .$$

The equations (4.3) thus become

$$
\begin{aligned}
e^{-2k\eta}u_{\sigma} &= -f_{\eta}p_{\sigma} + f_{\sigma}p_{\eta} \\
\varepsilon e^{-2k\eta}(uf_{\sigma\sigma} + u_{\sigma}f_{\sigma}) &= \varepsilon e^{-2k\eta}f_{\eta} - \ell e^{-2k\eta}f_{\eta} - p_{\eta} \qquad (4.4) \\
uf_{\eta} &= 1 \quad .
\end{aligned}
$$

These equations must hold for

$$-\infty < \sigma < \infty \;; \qquad 0 < \eta < 1$$

and the boundary conditions are

$$
\begin{aligned}
f(\sigma, 0) &= 0 \\
p(\sigma, 1) &= 0 \quad .
\end{aligned} \qquad (4.5)
$$

We assume now that all quantities in the new variables can be expanded in integral powers of ε, that is, we assume

$$
\begin{aligned}
f(\sigma, \eta, \varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k f_k(\sigma, \eta) \\
u(\sigma, \eta, \varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k u_k(\sigma, \eta) \qquad (4.6) \\
p(\sigma, \eta, \varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k p_k(\sigma, \eta)
\end{aligned}
$$

where the k^{th} order approximations are subject to the boundary conditions

$$f_k(\sigma,0) = 0 \tag{4.7}$$
$$p_k(\sigma,1) = 0 \quad .$$

As a consequence of the assumed expansions (4.6), the vertical velocity must have the expansion

$$v = \frac{\bar{v}}{c} = \frac{\bar{u}\bar{f}_x}{c} = uf_\xi = \sqrt{\varepsilon}\ uf_\sigma$$
$$= \sqrt{\varepsilon}\ [u_o f_{o\sigma} + \varepsilon(u_1 f_{o\sigma} + u_o f_{1\sigma}) + \dots\ ] \ .$$

We intend now to solve the equations which the lower order approximations must satisfy, and which come from substituting (4.6) in (4.4) and equating coefficients of like powers of ε. The boundary conditions (4.7) are set in the same way by substitution of (4.6) in (4.5). The expectation is that the analysis will show automatically that λ must correspond to one of the infinite number of critical speeds defined by the linear theory of the previous section.

The equations for the $zero^{th}$ order approximation are

$$\begin{aligned} e^{-2k\eta} u_{o\sigma} &= -f_{o\eta} p_{o\sigma} + f_{o\sigma} p_{o\eta} \\ 0 &= \lambda e^{-2k\eta} f_{o\eta} + p_{o\eta} \\ u_o f_{o\eta} &= 1 \end{aligned} \tag{4.8}$$

subject to

$$f_o(\sigma,0) = 0$$
$$p_o(\sigma,1) = 0 \ .$$

This is a nonlinear system and it is not easy to find its general solution. We can proceed, however, without knowing the general solution. Since we are interested in solitary waves which move in a medium initially at rest, we can assume that the dimensionless zeroth order approximation (with respect to the moving axes) must express a state of parallel flow with constant velocity $u_o = 1$. The solution which expresses this state is readily seen to be

$$f_o = \eta$$
$$u_o = 1$$
$$p_o = \frac{\ell}{2k} (e^{-2k\eta} - e^{-2k}) .$$

With this zeroth order approximation the equations for the first order approximation are

$$(4.9) \qquad \begin{aligned} e^{-2k\eta} u_{1\sigma} &= -\ell e^{-2k\eta} f_{1\sigma} - p_{1\sigma} \\ 0 &= e^{-2k\eta} - \ell e^{-2k\eta} f_{1\eta} - p_{1\eta} \\ f_{1\eta} + u_1 &= 0 \end{aligned}$$

subject to

$$f_1(\sigma,0) = 0$$
$$p_1(\sigma,1) = 0 .$$

This is a linear system which can be solved with little effort. The elimination of u_1 and p_1 from (4.9) gives

$$(4.10) \qquad f_{1\sigma\eta\eta} - 2kf_{1\sigma\eta} + 2k\ell f_{1\sigma} = 0 .$$

The first order approximation, f_1, to the stream line function must satisfy this equation in the strip $-\infty < \sigma < \infty$; $0 < \eta < 1$; and the boundary conditions

(4.11) $$f_{1\sigma}(\sigma,0) = 0$$

(4.12) $$f_{1\sigma\eta}(\sigma,1) = \lambda f_{1\sigma}(\sigma,1) \ .$$

The solution of (4.10) which satisfies (4.11) is

(4.13) $$f_{1\sigma} = a_1'(\sigma)e^{k\eta} \sin \alpha\eta$$

where $a_1'(\sigma)$ is arbitrary and $\alpha \neq 0$ is defined by

(4.14) $$\alpha = \sqrt{2k\lambda-k^2} \quad .$$

(The possibility of having $\alpha = 0$ can be discarded because this possibility implies a state of constant density, the analysis of which is covered in Part I.) From (4.13) integration with respect to σ yields

(4.15) $$f_1 = a_1(\sigma)e^{k\eta} \sin \alpha\eta+b_1(\eta)$$

where $b_1(\eta)$ is an arbitrary function of η. Since we confine our attention to solitary waves, we impose here the condition

$$f_1(-\infty,\eta) = 0 \quad .$$

This fixes $b_1(\eta)$. In fact, it permits us to take $b_1(\eta) \equiv 0$ since we choose $a_1(-\infty) = 0$. We now have

(4.16) $$f_1 = a_1(\sigma)e^{k\eta}\sin \alpha\eta .$$

[If we wanted to study cnoidal waves we would require f_1 to be periodic with some period T_1, and we would also require that $\int_{\sigma}^{\sigma+T_1} f_1(\sigma,\eta)d\sigma = 0$. This would fix $b_1(\eta)$ in (4.15) and permit us to write

$$f_1 = a_1(\sigma)e^{k\eta}\sin \alpha\eta$$

where $a_1(\sigma+T_1) = a_1(\sigma)$ and $\int_{\sigma}^{\sigma+T_1} a_1(\sigma)d\sigma = 0$.] Now, in order to satisfy the remaining boundary condition we find by substituting (4.16) in (4.12) that we must have

(4.17) $$a_1(\sigma)[(k-\ell)\tan \alpha+\alpha] = 0 .$$

It is at this point that the bifurcation phenomenon appears for the case of variable density. We find a solution which gives more than just a state of parallel flow if we take $a_1(\sigma) \not\equiv 0$ and choose ℓ so that it satisfies

$$(k-\ell)\tan \alpha+\alpha = 0$$

or what is the same thing

(4.18) $$\tan \alpha = \frac{2k\alpha}{\alpha^2-k^2} .$$

The bifurcation values of λ, in other words, the values of λ which satisfy (4.18), are just the values $\lambda_n = gh/c_n^2$ which satisfy (3.8) and give the critical speeds obtained from the linear theory.

The equation (4.18) is equivalent to the two equations

$$\tan \frac{\alpha}{2} = \frac{k}{\alpha}$$

$$-\cot \frac{\alpha}{2} = \tan \left(\frac{\alpha-\pi}{2}\right) = \frac{k}{\alpha} \quad .$$

Hence the roots of (4.18) are given by the intersections of the hyperbola

$$\beta = \frac{k}{\alpha}$$

with the curves

$$\beta = \tan \frac{\alpha}{2}$$
$$\beta = \tan \left(\frac{\alpha-\pi}{2}\right)$$

These curves are shown in Fig. 4.1. We see from this figure that

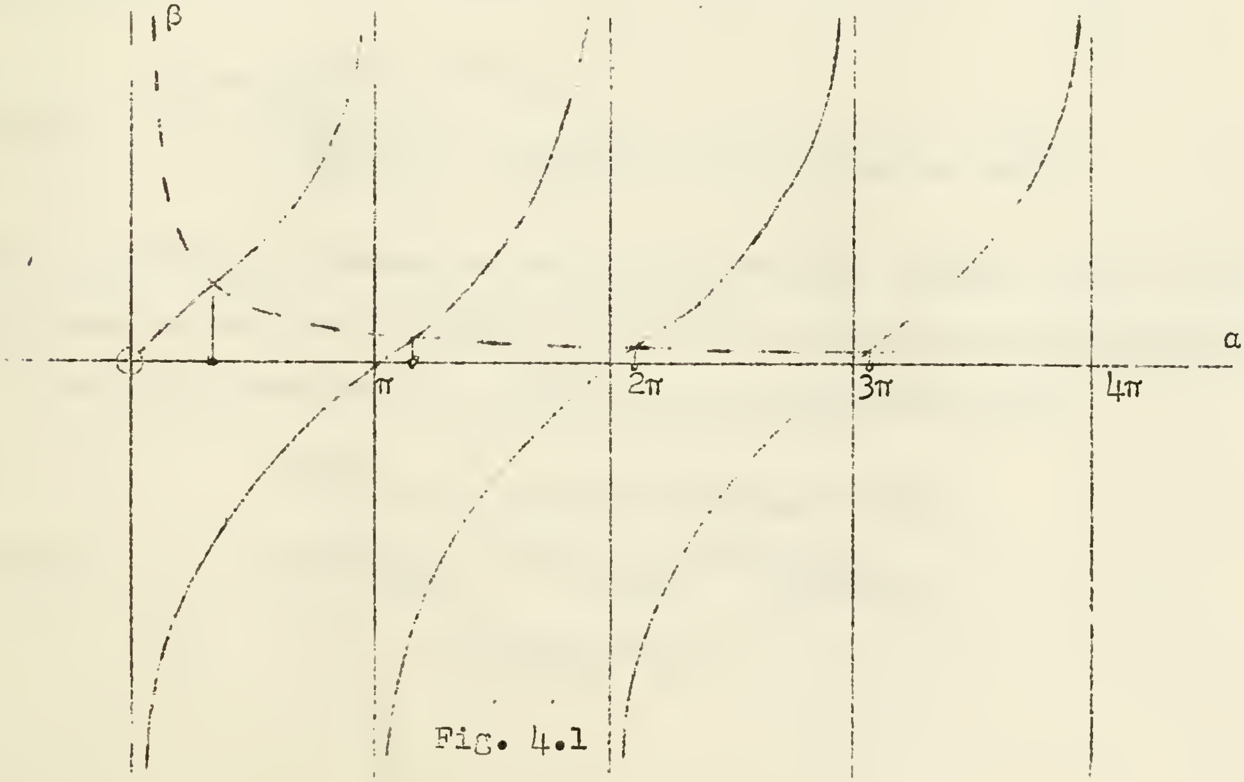

Fig. 4.1

there is a denumerable set of bifurcation values which increase in magnitude. If k is small these values are approximately integral multiples of π. Stated in terms of the critical speeds of the linear theory there is a discrete infinite set of critical speeds which has the value zero for its only limit point. We suppose hereafter, of course, that ℓ is a bifurcation value.

The first order approximation to the horizontal velocity (from the third of equations (4.9)) is given by

$$(4.19)\qquad \begin{aligned} u_1 &= - f_{1\eta} \\ &= -a_1(\sigma)e^{k\eta}\,[k \sin \alpha\eta+\alpha \cos \alpha\eta] \end{aligned}$$

and the first order approximation to the pressure (from the second of equations (4.9)) is obtained from an integration of

$$p_{1\eta} = e^{-2k\eta}-\ell e^{-2k\eta}f_{1\eta}$$

from which

$$(4.20)\qquad \begin{aligned} p_1 &= \frac{1}{2k}\,(e^{-2k}-e^{-2k\eta}) \\ &\quad - \frac{a_1(\sigma)}{2k}\, e^{-k}\ \ [(\alpha^2-k^2) \sin \alpha\eta -2k\alpha \cos \alpha\eta] \ . \end{aligned}$$

The function $a_1(\sigma)$ cannot be determined at this stage; to find it we must pass to the set of equations for the second order approximations.

The equations for the second order approximation are

$$(4.21)\qquad \begin{aligned} e^{-2k\eta}u_{2\sigma} &= f_{1\sigma}p_{1\eta}+f_{2\sigma}p_{0\eta}-p_{1\sigma}f_{1\eta}-p_{2\sigma} \\ e^{-2k\eta}f_{1\sigma\sigma} &= e^{-2k\eta}f_{1\eta}-\ell e^{-2k\eta}f_{2\eta}-p_{2\eta} \\ f_{2\eta}&+u_1f_{1\eta}+u_2 = 0 \end{aligned}$$

subject to the boundary conditions

$$f_2(\sigma,0) = 0$$

$$p_2(\sigma,1) = 0 \ .$$

By elimination of u_2 and p_2 from (4.21) and use of the first order equations, (4.21) leads to

$$\text{(4.22)} \qquad f_{2\sigma\eta\eta} - 2kf_{2\sigma\eta} + 2k\ell f_{2\sigma} = L(\sigma,\eta)$$

where

$$L(\sigma,\eta) = -f_{1\sigma\sigma\sigma} + 3f_{1\eta\eta}f_{1\sigma\eta}$$
$$+ 3f_{1\eta}(f_{1\sigma\eta\eta} - 2kf_{1\sigma\eta}) + 2kf_{1\sigma}$$

is a known function of $a_1(\sigma)$; and f_2 satisfies (4.22) in the strip $-\infty < \sigma < \infty$; $0 < \eta < 1$ with the boundary conditions

$$\text{(4.23)} \qquad f_2(\sigma,0) = 0 \ ;$$

$$\text{(4.24)} \qquad f_{2\sigma\eta}(\sigma,1) = \ell f_{2\sigma}(\sigma,1) + 3\ell f_{1\eta}(\sigma,1)f_{1\sigma}(\sigma,1) - f_{1\sigma}(\sigma,1) \ .$$

The equation for the determination of $a_1(\sigma)$ can be found by multiplying each side of (4.22) by $e^{-k\eta}\sin \alpha\eta$ and integrating the resulting products from zero to one. That is, if we integrate the left side of

$$\int_0^1 e^{-k\eta} \sin \alpha\eta [f_{2\sigma\eta\eta} - 2kf_{2\sigma\eta} + 2k\ell f_{2\sigma}]d\eta = \int_0^1 e^{-k\eta} \sin \alpha\eta \, L(\sigma,\eta)d\eta$$

by parts; use the boundary conditions (4.23), (4.24) and assume, of course, that α is a root of (4.18) we find that $a_1(\sigma)$ must satisfy

(4.25)
$$e^{-k} \sin \alpha[3\ell f_{1h}(\sigma,1)f_{1\sigma}(\sigma,1)-f_{1\sigma}(\sigma,1)] = \int_0^1 e^{-kh} \sin \alpha h L(\sigma,h)dh .$$

Then if we substitute $f_1 = a_1(\sigma)e^{kh} \sin \alpha h$ and perform the integration on the right hand side of (4.25) it turns out that $a_1(\sigma)$ must satisfy

(4.26)
$$m_o a_1'''(\sigma) = m_1 a_1(\sigma) a_1'(\sigma) + m_2 a_2'(\sigma) .$$

In this equation the quantities m_o, m_1, and m_2 are given by:

(4.27)
$$m_o = \frac{\ell^2}{\alpha^2} \left[\frac{k(k^2-\alpha^2)}{(k^2+\alpha^2)^2}\right] > 0$$
$$m_1 = \frac{3\ell\alpha(\alpha^2+k^2)}{(k^2-\alpha^2)(k^2+9\alpha^2)} [8(k^2+\alpha^2)e^k \cos \alpha - (k^2-\alpha^2)]$$
$$m_2 = \frac{(\alpha^2+k^2)}{4k\alpha^2} (\alpha^2+k^2+2k) > 0 .$$

The constant m_2 is always greater than zero, and it can be shown that this is also true for m_o. Since

$$\cos \alpha = \pm \frac{(k^2-\alpha^2)}{k^2+\alpha^2}$$

the constant m_1 may be greater than zero or less than zero but it cannot be zero since $8e^k \pm 1$ cannot be zero.

The equation (4.26) has the same form as the equation (4.20) in Part I which determines solitary and cnoidal waves for a medium of two layers. Cnoidal waves for our medium of continuously variable

density can be derived from the general solution of (4.26). Solitary waves result if we impose special boundary conditions on (4.26). One such condition has already been set, namely $a_1(-\infty) = 0$. For reasons explained in Part I we also impose the conditions $a_1'(-\infty) = 0$ and $a_1''(-\infty) = 0$. With these conditions, an integration of (4.26) gives

$$m_o a_1''(\sigma) = m_1 \frac{a_1^2(\sigma)}{2} + m_2 a_1(\sigma) \tag{4.28}$$

and another integration after multiplication of (4.28) by $a_1'(\sigma)$ gives

$$m_o [a_1'(\sigma)]^2 = \frac{m_1}{3} a_1^3(\sigma) + m_2 a_1^2(\sigma) \; . \tag{4.29}$$

Then, the solution of (4.29) which has zero slope at $\sigma = 0$ (conforming with the assumption that the vertical axis passes through the crest of the wave) is

$$a_1(\sigma) = -\frac{3m_2}{m_1} \operatorname{sech}^2 \frac{\sigma}{2} \sqrt{\frac{m_2}{m_o}} \; .$$

We suppose now that what we have found gives a sufficiently accurate representation of solitary waves in a medium with the assumed exponential variation in density. If we use the symbol $\cong$ to denote approximation; and if we return to the x, γ variables, a collection of the results shows the following:

$$\bar{f}(x,\gamma) \cong h - \frac{3m_2}{m_1}(\gamma-\lambda) h e^{k\eta} \sin a\eta \operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{m_2}{m_o}(\gamma-\lambda)} \tag{4.30}$$

$$\bar{u}(x,\gamma) \cong c + \frac{3m_2}{m_1}(\gamma-\lambda) c e^{k\eta}(k \sin a\eta + a \cos a\eta) \operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{m_2}{m_o}(\gamma-\lambda)} \tag{4.31}$$

$$(4.32)\quad \bar{p}(x,\gamma) \cong \delta_o c^2 \left\{ \frac{\lambda}{2k}(e^{-2k\eta} - e^{-2k}) + \frac{3m_2}{m_1}\frac{(\ell-\lambda)}{2k} e^{-k\eta}[(\alpha^2-k^2)\sin\alpha\eta - 2k\alpha\cos\alpha\eta] \cdot \cdot \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)} \right\}$$

$$(4.33)\quad \bar{v}(x,\gamma) \cong c(\ell-\lambda)^{\frac{3}{2}} \frac{3m_2}{m_1}\sqrt{\frac{m_2}{m_o}}\, e^{k\eta}\sin\alpha\eta \cdot \cdot \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)} \cdot \tanh \frac{x}{2h}\sqrt{\frac{m_2}{m_o}(\ell-\lambda)} \quad .$$

In these results

$$\eta = \frac{\gamma}{ch} \qquad\qquad \lambda = \frac{gh}{c^2}$$

where c is the velocity of the solitary wave; and $k > 0$ is the index of the exponential density distribution $\bar{\delta}(x,\gamma) = \delta_o e^{-2k\eta}$. In terms of the vertical distance, y_o, from the bottom of the channel at $x = -\infty$,

$$\gamma = \int_0^{y_o} \psi_y(-\infty, y)\,dy = \int_0^{y_o} c\,dy = cy_o \quad .$$

The constants m_i are given by (4.27) and ℓ is to be determined from

$$2k\ell = \alpha^2 + k^2$$

where α is a non zero positive root of

$$\tan\alpha = \frac{2k\alpha}{\alpha^2 - k^2} \quad .$$

Since $\frac{m_2}{m_o}(\ell-\lambda)$ appears under the square root sign and since $\frac{m_2}{m_o} > 0$, it follows that $\ell-\lambda$ must be positive, which means that the speed c of the wave is always greater than the critical speed.

5. Properties of the Solitary Waves. Solitary Waves whose Speeds are near the two Highest Critical Speeds.

In this section we discuss some of the properties of the solitary waves which are obtained from (4.30) when k is small. Particular attention is given to the solitary waves which travel with speeds near the two highest critical speeds.

The stream lines throughout the medium are given by

$$(5.1)\quad y = \bar{f}(x,\gamma) \cong h\hbar - \frac{3m_2}{m_1}(\ell-\lambda)he^{k\hbar}\sin\alpha\hbar\ \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{m_2}{m_0}(\ell-\lambda)}$$

where $0 \leq \hbar \leq 1$ and where α may be any one of the values determined by

$$(5.2)\qquad \tan\frac{\alpha}{2} = \frac{k}{\alpha}$$

or

$$(5.3)\qquad \tan\left(\frac{\alpha-\pi}{2}\right) = -\cot\frac{\alpha}{2} = \frac{k}{\alpha}\ .$$

When k is small the values of α which satisfy (5.2) are slightly larger than even multiples of π, and the values of α which satisfy (5.3) are slightly larger than odd multiples of π. We proceed to discuss these two cases separately.

Case I. $\alpha \cong m2\pi$; but $\alpha > m2\pi$ (m an integer).

For a value of α which is nearly a multiple of 2π we have

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{\alpha^2-k^2}{\alpha^2+k^2}$$

and hence

$$-\frac{3m_2}{m_1} = \frac{(\alpha^2+k^2+k)(\alpha^2-k^2)(k^2+9\alpha^2)}{2\alpha^3(\alpha^2+k^2)[8(k^2+\alpha^2)e^k \cos\alpha-(k^2-\alpha^2)]}$$

$$= \frac{(\alpha^2+k^2+k)(k^2+9\alpha^2)}{2\alpha^3(\alpha^2+k^2)(8e^k+1)}$$

which is positive. If we write

$$q = -\frac{3m_2}{m_1}(\ell-\lambda) > 0 ,$$

the stream lines are given by

$$(5.4)\quad y = \bar{f}(x,\gamma) \cong h\eta+qhe^{k\eta}\sin\alpha\eta\,\mathrm{sech}^2\frac{x}{2h}\sqrt{q\left(\frac{-m_1}{3m_0}\right)}$$

where

$$(5.5)\quad -\frac{m_1}{3m_0} = \frac{4k\alpha^3(k^2+\alpha^2)^2(8e^k+1)}{(k^2+9\alpha^2)[2k(k^2-\alpha^2)+(k^2+\alpha^2)^2]} .$$

In terms of q, α and k the speed of the wave is given by

$$\lambda = \ell + \frac{m_1}{3m_2}q$$

or

$$(5.6)\quad \frac{gh}{c^2} = \frac{\alpha^2+k^2}{2k}\left[1-\frac{4k\alpha^3(8e^k+1)q}{(\alpha^2+k^2+2k)(k^2+9\alpha^2)}\right] .$$

From (5.4) we see that the stream lines are parallel to the horizontal bottom of the channel for those values of η which satisfy $\sin\alpha\eta = 0$. If α is between zero and π there is no stream line, other than the bottom one, which is horizontal; but if $\alpha \cong m2\pi$ where m is a positive integer, the horizontal stream lines occur when η is approximately equal to

$$\eta \doteq \frac{n}{2m}$$

where n and $m \neq 0$ are integers such that $0 \leq \frac{n}{2m} \leq 1$.

The maximum departure of the stream lines from $y = h\eta$ occurs at $x = 0$. When α is between 0 and π the maximum departure occurs at the free surface and the behavior is like the ordinary solitary wave in a medium of constant density. Otherwise, the extrema of the stream lines with respect to $y = h\eta$ occur at $x = 0$, and the values of which satisfy

$$k \sin \alpha\eta + \alpha \cos \alpha\eta = 0 ,$$

or (since $\frac{k}{\alpha} = \tan \frac{\alpha}{2}$)

$$\cos \alpha(\eta - \tfrac{1}{2}) = 0 .$$

We denote these values by η_i. They are approximately equal to

$$\eta_i = \frac{2i+1}{4m}$$

where m is a positive integer and $i = 0,1,2,\ldots,2m-1$. The maxima and minima of the stream lines with respect to $y = h\eta$ are given approximately by

$$qhe^{k\eta_i} \sin \alpha\eta_i = qhe^{k(\frac{2i+1}{m})}(-1)^i .$$

The extremum nearest the bottom is a positive maximum; and the extrema, with slightly increasing absolute values, alternate between positive and negative values as we ascend from the bottom of the channel to the top. For obvious physical reasons, an extremum must be less in absolute value than the distance from its place of occurrence, $(0,\eta_i)$, to the nearest stream line parallel to the bottom.

Hence when α is large the amplitudes of the internal waves and the surface wave must be small. Approximately speaking, the stream lines in the strip $0 < \eta < \frac{1}{2m}$ are lines of elevation above $y = h\eta$. The lines in the next strip, $\frac{1}{2m} < \eta < 2\cdot\frac{1}{2m}$ are lines of depression below $y = h\eta$, and so on to the free surface. The profile of the free surface is given by

$$y = h+qhe^{k} \sin \alpha \ \mathrm{sech}^2 \frac{x}{2h}\sqrt{q(\frac{-m_1}{3m_o})} \quad .$$

The free surface is a wave of elevation, but if α is not between 0 and π the amplitude tends to be quite small since $\sin \alpha \cong \sin 2\cdot m\pi = 0$.

Let us denote the least positive root of

$$\tan \alpha = \frac{k}{\alpha}$$

by α_1. For k small, α_1 is given approximately by

$$\alpha_1^2 \cong 2k - \frac{k^2}{3} \ .$$

For this root the stream lines are lines of elevation above $y = h\eta$, and the maximum amplitude occurs at the free surface. The maximum amplitude is

$$q_1 = qhe^{k} \sin \alpha_1 \cong qhe^{k} \sin \sqrt{2k} \quad .$$

A numerical calculation in which we ultimately retain only the first power of k, shows from (5.6) that the wave speed is approximately given by

$$\text{(5.7)} \qquad \frac{c^2}{gh} \cong \frac{1}{1+\frac{k}{3}} \left(1+ \frac{q_1}{h}\right) .$$

A similar calculation for $-m_1/3m_o$ shows

$$- \frac{m_1}{3m_o} \cong 3\sqrt{2k} \quad .$$

Hence the stream lines for $\alpha = \alpha_1$ are defined by

$$(5.8) \quad \overline{f}(x,\gamma) \cong h\eta + q_1 e^{k(\eta-1)} \frac{\sin\sqrt{2k}\eta}{\sin\sqrt{2k}} \cdot \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3\sqrt{2k}\, q_1}{he^k \sin\sqrt{2k}}} \quad .$$

Notice that if we take the limit of the right hand side of (5.8) as $k \to 0$ we have

$$\overline{f}(x,\gamma) \cong h\eta + q_1\eta \operatorname{sech}^2 \frac{x}{2h}\sqrt{\frac{3q_1}{h}}$$

which coincides with the known result for the stream lines in a medium of constant density. Also, as $k \to 0$, the formula (5.7) becomes

$$\frac{c^2}{gh} \cong 1 + \frac{q_1}{h}$$

which is the Scott Russell formula.

Case II. $\alpha \cong (2m+1)\pi$; but $\alpha > (2m+1)\pi$ $(m = 0, 1, 2\ldots)$.

For a value of α which is nearly an odd multiple of π we have

$$\cos\alpha = \frac{\cot^2\frac{\alpha}{2} - 1}{\cot^2\frac{\alpha}{2} + 1} = \frac{k^2-\alpha^2}{k^2+\alpha^2}$$

and

$$- \frac{3m_2}{m_1} = \frac{(\alpha^2+k^2+2k)(\alpha^2-k^2)(k^2+9\alpha^2)}{2\alpha^3(\alpha^2+k^2)[8(k^2+\alpha^2)e^k \cos\alpha-(k^2-\alpha^2)]}$$

$$= - \frac{(\alpha^2+k^2+2k)(k^2+9\alpha^2)}{2\alpha^3(\alpha^2+k^2)(8e^k-1)}$$

which is negative. If we write

$$Q = \frac{3m_2}{m_1}(\ell-\lambda) > 0 \ ,$$

the stream lines are given by

$$(5.9) \qquad y = \overline{f}(x,\gamma) \cong h\eta - Qhe^{k\eta} \sin \alpha\eta \ \mathrm{sech}^2 \frac{x}{2h}\sqrt{Q(\frac{m_1}{3m_0})}$$

where

$$\frac{m_1}{3m_0} = \frac{4k\alpha^3(k^2+\alpha^2)^2[8e^k-1]}{(k^2+9\alpha^2)[2k(k^2-\alpha^2)+(k^2+\alpha^2)^2]} \ .$$

In terms of Q, α, and k the speed of the wave is given by

$$\lambda = \ell - \frac{m_1}{3m_2} Q$$

or

$$(5.10) \qquad \frac{gh}{c^2} = \frac{\alpha^2+k^2}{2k}\left[1 - \frac{4k\alpha^3(8e^k-1)Q}{(\alpha^2+k^2+2k)(k^2+9\alpha^2)}\right.$$

From (5.9) we see that the stream lines are parallel to the horizontal bottom of the channel for those values of η which satisfy $\sin \alpha\eta = 0$. These horizontal stream lines occur when η is approximately equal to

$$\eta \cong \frac{n}{2m+1}$$

where n takes the values $n = 0,1,2,\ldots,2m+1$.

The extrema of the stream lines with respect to $y = h\eta$ occur at $x = 0$ and those values of η which satisfy

$$k \sin \alpha\eta + \alpha \cos \alpha\eta = 0 \ .$$

Since for the case under consideration $-\cot \frac{\alpha}{2} = \frac{k}{\alpha}$, these values of η satisfy

$$\sin \alpha(\eta - \tfrac{1}{2}) = 0 \ .$$

We will denote these values by η_j. They are approximately equal to

$$\eta_j \cong \frac{2j+1}{2(2m+1)}$$

when $j=0,1,2,\ldots,2m$. The maxima and minima of the stream lines with respect to $y = h\eta$ are given approximately by

$$-Qhe^{k\eta_j} \sin\alpha\eta_j \cong -(-1)^j \, Qhe^{\frac{k(2j+1)}{2(2m+1)}} \ .$$

The extremum nearest the bottom is a negative minimum; and the extrema with slightly increasing absolute values alternate between negative and positive values as we ascend from the bottom of the channel to the top. The absolute value of an extremum must be less than the distance from $(0,\eta_j)$ to the nearest horizontal stream line. This physical condition imposes an upper bound for Q. The amplitude of the internal waves must tend to zero as α increases.

For the strip approximately given by $0 < \eta < \frac{1}{2m+1}$, the stream lines are lines of depression below $y = h\eta$. In the next strip $\frac{1}{2m+1} < \eta < \frac{2}{2m+1}$ the stream lines are lines of elevation above $y = h\eta$. The alternation of strips of lines of depression and strips of lines of elevation continues until the free surface is included. The profile of the wave at the free surface is given by

$$y = h - Qhe^k \sin \alpha \ \mathrm{sech}^2 \frac{x}{2h} \sqrt{Q\left(\frac{m_1}{3m_o}\right)} \ .$$

This is a wave of elevation because α is slightly larger here than an odd multiple of π and hence $\sin \alpha$ is negative, but since $\sin \alpha$ is small the amplitude of the wave tends to be small.

As a specific example of a solitary wave of the kind we are now considering, let us take the solitary wave which corresponds to the least value of α which satisfies

$$-\cot \frac{\alpha}{2} = \frac{k}{\alpha} \quad .$$

We will denote this value by α_2. When k is small, α_2 is approximately

$$\alpha_2 \cong \pi + \frac{2k}{\pi} \quad .$$

For this value of α, a calculation of $\frac{m_1}{3m_o}$, in which we retain only the first power of k, gives

$$\frac{m_1}{3m_o} = \frac{28k\pi}{9} \quad ,$$

and the stream lines are given by

$$y = \bar{f}(x,\gamma) = h - Qhe^{k\eta} \sin \alpha_2\eta \, \mathrm{sech}^2 \frac{x}{2h}\sqrt{\frac{Q28k\pi}{9}} \quad .$$

We see that there is a horizontal stream line when η is such that

$$\sin \alpha_2\eta = 0 \; .$$

This value of η is approximately

$$\eta = \frac{\pi}{\alpha_2} \cong \frac{\pi}{\pi+\frac{2k}{\pi}} \cong 1 - \frac{2k}{\pi^2} \quad .$$

Thus the horizontal stream line is close to the free surface. In the strip $0 < \eta < \frac{\pi}{\alpha_2}$ the stream lines are lines of depression below $y = h\eta$. In the narrow strip $\frac{\pi}{\alpha_2} < \eta < 1$, the stream lines are lines of elevation above $y = h\eta$. The stream line of greatest depression below $y = h\eta$ is given by the value of η which satisfies

$$\sin \alpha_2(\ell - \tfrac{1}{2}) = 0 ,$$

namely $\ell = 1/2$. The maximum departure of this stream line from $y = \frac{h}{2}$ is $Qhe^{k/2} \sin \frac{\alpha_2}{2} = Q_2$ say. The maximum amplitude of the wave at the free surface is

$$-Qhe^{k}\sin \alpha_2 = -Q_2 \; 2e^{\frac{k}{2}} \cos \frac{\alpha_2}{2} .$$

This is approximately $2Q_2 \; e^{\frac{k}{2}} \sin \frac{k}{\pi}$ which is small compared with Q_2.

The approximate speed of the solitary wave can be obtained from (5.10). In terms of Q_2 we have

$$\frac{gh}{c^2} = \frac{\alpha_2^2+k^2}{2k} = [1 - \frac{4k\alpha_2^3(8e^{k}-1)}{(\alpha_2^2+k^2+2k)(k^2+9\alpha_2^2)} \cdot \frac{Q_2}{he^{\frac{k}{2}} \sin \frac{\alpha_2}{2}}] .$$

If we perform the calculation and retain only the first powers of k we find

$$\frac{c^2}{gh} \cong \frac{2k}{\pi^2} [1 + \frac{28k}{9\pi} \frac{Q_2}{h}] .$$

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